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Abstract

Hairer's regularity structures transformed the solution theory of singular stochastic partial differential equations. The notions of positive and negative renormalisation are central and the intricate interplay between these two renormalisation procedures is captured through the combination of cointeracting bialgebras and an algebraic Birkhoff-type decomposition of bialgebra morphisms. This work revisits the latter by defining Bogoliubov-type recursions similar to Connes and Kreimer's formulation of BPHZ renormalisation. We then apply our approach to the renormalisation problem for SPDEs as well as the proposal for resonance based numerical schemes for certain partial differential equations in numerical analysis introduced in a recent work by the first author.

Contents

1 Introduction

The theory of Regularity Structures (RS) has been developed to its full generality within a few years since its initial presentation by Hairer [Hai14]. Thanks to recent progress, one has local well-posedness results for a large class of singular Stochastic Partial Differential equations (SPDEs). This achievement relies, among others, on the following papers [BHZ19, CH16, BCCH17]. It culminated in the construction of a natural random dynamic on the space of loops in a Riemannian manifold described in [BGHZ19]. Friz and Hairer [FH14] present a textbook introduction to RS and [BHZ20, BH20] give short respectively extended surveys on these developments. The algebraic foundation of the theory has been developed in [BHZ19], where two renormalisation procedures are shown to be in cointeraction: the first recenters distributions around a point such that they can be understood as recentered monomials. The second renormalisation cures divergences coming from ill-defined distributional products in a singular SPDE. In the abstract of reference [BHZ19], the key parts of these renormalisations have been highlighted: "Two twisted antipodes play a fundamental role in the construction and provide a variant of the algebraic Birkhoff factorisation".

The main contribution of this paper is to make this link more precise and to explore the extend to which Bogoliubov's recursions and hence the algebraic Birkhoff factorisation are altered in the context of the renormalisation problem of SPDEs. We will also indicate how this approach appeared recently in the context of local error analysis in numerical analysis [BS20]. In this context, several algebraic simplifications have been observed.

Let us outline the paper by summarising the content of the sections. In Section 2, we recall the algebraic Birkhoff factorisation following Connes and Kreimer [CK00], which unveiled an elegant group theoretical formulation of the BPHZ renormalisation procedure in perturbative quantum field theory [BP57, Hep69, Zim69]. We refer the reader to [CK82, Col84, Zav90, Pan12] for useful references on renormalisation in perturbative quantum field theory. Bogoliubov's recursions for counterterms and renormalised amplitudes are characterised as solutions of a factorisation problem in the group of characters over a (specific Feynman graph or rooted tree) Hopf algebra. Then we introduce a factorisation-type renormalisation with the coproduct replaced by a coaction. The counterterm recursion is defined via a comodule structure. These two structures are considered for a connected Hopf algebra. We introduce also Taylor-jet operators forming a (family of) Rota–Baxter map(s), which is central for the Bogoliubov recursions to solve the factorisation problem. Section 3 contains the new and important results. We consider decorated trees as they appear in regularity structures and define a comodule-Hopf algebra on them. This structure has been originally introduced in [BHZ19]. We present its algebraic construction and postpone the link to SPDEs to the next section. The main difficulty lies in the fact that the Hopf algebra at play is not connected. Therefore, the results presented in Section 2 cannot be applied directly if one wants to set up a Birkhoff-type factorisation. This problem is circumvented by defining a modified reduced coproduct and use a family of Rota–Baxter maps in order to give one of the main definitions of this paper (Definition 3.11). Then, by exploiting the Rota–Baxter property one can show the main result (Theorem 3.13). It is also shown that under certain assumptions, the recentering map does not depend on a priori recentering of the polynomials. These result have to be understood as an alternative way of defining the notion of model, which is a critical part in Hairer's theory [Hai14]. In Section 4, we present three applications of the results from Section 3. The first one concerns the construction of the recentering map. Then, we present the notion of negative renormalisation, which is close in spirit to the approach outlined in Section 2, but where the comodule structure is used instead. The last subsection is devoted to a rather recent progress which appeared in [BS20]. It proposes the use of Birkhoff-type factorisation in numerical analysis. It is again based mainly on the formalism given in Section 2. In fact, thanks to some simplifications, one can work with a connected Hopf algebra.

2 Algebraic Birkhoff factorisation

Connes and Kreimer discovered a Hopf algebraic formulation of renormalisation in perturbative quantum field theory [CK00]. It permits to formulate the so-called BPHZ subtraction method [Col84, Zav90] in terms of an algebraic Birkhoff decomposition of Feynman rules seen as an element in the group of Hopf algebra characters. The factors in this decomposition give the renormalised respectively counterterm parts.

The notion of renormalisation in regularity structures permits as well a Hopf algebraic formulation. However, the tree Hopf algebras at play are not necessarily connected. Moreover, renormalisation can not be described as decomposition at a group theoretical level. Instead, we will have to consider a variant of this approach using a comodule structure.

Let $H = \mathbf{C}\mathbf{1} \oplus \bigoplus_{n>0} H^{(n)}$ be a connected graded Hopf algebra over $\mathbf{C}$ with coproduct $\Delta : H \to H \otimes H$, antipode $\mathcal{A} : H \to H$ and counit $\mathbf{1}^* : H \to \mathbf{C}$. Recall that the latter is a linear map which equals 1 on $\mathbf{1}$ and zero else. In the sequel, we will use Sweedler's notation in order to describe the coproduct Δ as well as the corresponding reduced coproduct, Δ':

$$\begin{aligned}\Delta\tau = \sum_{(\tau)} \tau^{(1)} \otimes \tau^{(2)} &= \Delta'\tau + \tau \otimes \mathbf{1} + \mathbf{1} \otimes \tau \\ &= \sum_{(\tau)}{}' \tau' \otimes \tau'' + \tau \otimes \mathbf{1} + \mathbf{1} \otimes \tau.\end{aligned}$$

Remark 2.1 Let $\mathcal{RT}$ denote the set of non-planar rooted trees and $\mathcal{T} := \langle \mathcal{RT} \rangle$ is the corresponding space. A tree is naturally graded by its number of vertices. The Butcher–Connes–Kreimer Hopf algebra of rooted trees, $H_{\mathcal{T}}$, provides a key example of a connected graded Hopf algebra – of combinatorial nature. It plays an important role in numerical analysis in relation to Butcher's B-series [CHV10]. Connes and Kreimer studied $H_{\mathcal{T}}$ in great detail in the context of renormalisation in perturbative quantum field theory [CK98]. They defined its coproduct using the notion of admissible cuts on trees. However, they also described a recursive formula based on the fact that any rooted tree $\tau \in \mathcal{T}$, different from the empty tree, **1**, can be written in terms of the B_+-operator. Indeed, in terms of the latter, we have that $\tau = B_+(\tau_1 \cdots \tau_n)$, which connects the roots of the trees in the forest $\tau_1 \cdots \tau_n \in H_{\mathcal{T}}$ to a new root. The coproduct on $H_{\mathcal{T}}$ then satisfies the relation

$$\Delta_{\text{CK}}(\tau) = \mathbf{1} \otimes \tau + (B_+ \otimes \text{id})\Delta_{\text{CK}}(\tau_1 \cdots \tau_n). \tag{2.1}$$

We denote by char(H, A) the set of characters from the Hopf algebra H into a commutative unital **C**-algebra, A. These are linear algebra morphisms forming a group with respect to the convolution product

$$\psi \star \varphi := m_A(\psi \otimes \varphi)\Delta \qquad H \xrightarrow{\Delta} H \otimes H \xrightarrow{\psi\otimes\varphi} A \otimes A \xrightarrow{m_A} A. \tag{2.2}$$

The convolution inverse of a character, $\varphi \in$ char(H, A), is given through composition with the antipode, $\varphi^{-1} = \varphi \circ \mathcal{S}$, and the unit for the convolution product is the co-unit $\mathbf{1}^*$ [1].

It is furthermore assumed that a linear projection, $Q : A \to A$, is defined on A, which satisfies the (weight one) Rota–Baxter identity:

$$Q(f) \cdot_A Q(g) + Q(f \cdot_A g) = Q(Q(f) \cdot_A g + f \cdot_A Q(g)) \tag{2.3}$$

for any $f, g \in A$. Here, $f \cdot_A g := m_A(f \otimes g)$ denotes the product of f and g in the algebra A. The associated projector, $\tilde{Q} :=$ id $- Q$, also satisfies identity (2.3). As a result, A splits into two subalgebras $A_- := Q(A)$ and $A_+ := \tilde{Q}(A)$:

$$A = A_- \oplus A_+.$$

Remark 2.2 One of the main examples is given by the algebra of Laurent series, $A = \mathbf{C}[[t, t^{-1}]$, with finite pole-part. In this context $A_- = t^{-1}\mathbf{C}[t^{-1}]$ and $A_+ = \mathbf{C}[[t]]$, such that Q keeps only the pole part of a series:

$$Q\Big(\sum_n a_n t^n\Big) = \sum_{n<0} a_n t^n \in A_-.$$

Connes and Kreimer discovered a Hopf algebraic approach to the so-called BPHZ renormalization method in perturbative quantum field theory [CK00]. It permits the formulation of the process of perturbative renormalisation in terms of a factorisation theorem for –regularised– Hopf algebra characters. We now briefly recall this so-called algebraic Birkhoff decomposition.

Proposition 2.3 *For every character $\varphi \in$ char(H, A), there exist unique algebra morphisms $\varphi_- : H \to A_-$ and $\varphi_+ : H \to A_+$ defined in terms of the recursions*

$$\varphi_- = \mathbf{1}^* - Q((\varphi - \mathbf{1}^*) \star \varphi_-) \tag{2.4}$$

[1] For the sake of notational transparency, we'll suppress here the unit-map that should follow the co-unit.

$$\varphi_+ = \mathbf{1}^* + \tilde{Q}((\varphi - \mathbf{1}^*) \star \varphi_-) \tag{2.5}$$

and yielding the algebraic Birkhoff factorisation

$$\varphi_+ \star \varphi_-^{-1} = \varphi. \tag{2.6}$$

Definition 2.4 The map $\bar{\varphi} = (\varphi - \mathbf{1}^*) \star \varphi_-$ is called Bogoliubov's preparation map. The maps φ_- and φ_+ are called counterterm respectively renormalised character.

Remark 2.5 We note that $\varphi_+^{-1} = \varphi_+ \circ \mathscr{A}$ can be computed recursively

$$\varphi_+^{-1} = \mathbf{1}^* - \tilde{Q}(\varphi_+ \star (\varphi - \mathbf{1}^*)).$$

Identity (2.3) then implies that $\varphi_+^{-1} \star \varphi_- = \mathbf{1}^* - \varphi_+^{-1} \star (\varphi - \mathbf{1}^*) \star \varphi_-$, from which (2.6) follows immediately.

Observe that $\varphi_+ = \varphi \star \varphi_- = m_A(\varphi \otimes \varphi_-)\Delta$ and evaluating on a $\tau \in H$ different from $\mathbf{1}$ yields the explicit formulas

$$\begin{aligned}
\varphi_-(\tau) &= \mathbf{1}^*(\tau) - Q((\varphi - \mathbf{1}^*) \star \varphi_-)(\tau) = -Q(\bar{\varphi}(\tau)) \\
\bar{\varphi}(\tau) &= \varphi(\tau) + \sum_{(\tau)}{}' \varphi(\tau') \cdot_A \varphi_-(\tau'') \\
\varphi_+(\tau) &= \varphi(\tau) + \sum_{(\tau)}{}' \varphi(\tau') \cdot_A \varphi_-(\tau'') + \varphi_-(\tau) = \tilde{Q}(\bar{\varphi}(\tau)).
\end{aligned} \tag{2.7}$$

Remark 2.6 Note that when $Q = \mathrm{id}_A$, one recovers the recursive definition for the antipode $\mathscr{A}$. Indeed, from (2.4) we deduce that

$$\varphi_- = \mathbf{1}^* - (\varphi - \mathbf{1}^*) \star \varphi_- = \frac{1}{\mathbf{1}^* + (\varphi - \mathbf{1}^*)} = \varphi(\mathrm{id}^{-1}) = \varphi \circ \mathscr{A},$$

which is consistent with the antipode being the convolution inverse of the identity map, implying the antipode recursion (thanks to the connectedness of H)

$$\mathscr{A}\tau = -\tau - \sum_{(\tau)}{}' \tau' \mathscr{A}\tau'' = -\tau - \sum_{(\tau)}{}' \mathscr{A}(\tau')\tau''.$$

In the next section, we consider $\hat{H}$ being a right-comodule over H. The coaction

$$\hat{\Delta} : \hat{H} \to \hat{H} \otimes H$$

is used to built a variant of factorisation (2.6). We suppose also given an injection $\iota : H \to \hat{H}$.

Proposition 2.7 *For every $\varphi \in$ char$(\hat{H}, A)$, there are unique linear maps $\varphi_- : H \to A_-$ and $\varphi_+ : \hat{H} \to A$ such that for every $\tau \in H$:*

$$\begin{aligned} \varphi_- &= \mathbf{1}^* - Q \circ \bar{\varphi} \circ \iota, \quad \bar{\varphi} = m_A((\varphi - \mathbf{1}^*) \otimes \varphi_-)\hat{\Delta} \\ \varphi_+ &= \varphi \star \varphi_- = m_A(\varphi \otimes \varphi_-)\hat{\Delta}, \end{aligned} \tag{2.8}$$

where the reduced co-module map

$$\hat{\Delta}' \circ \iota(\tau) = \sum_{(\iota(\tau))}' \iota(\tau)' \otimes \iota(\tau)'' = \hat{\Delta} \circ \iota(\tau) - \iota(\tau) \otimes \mathbf{1} - \mathbf{1} \otimes \iota(\tau)$$

corresponds to the reduced coaction. The linear maps φ_- and φ_+ are also algebra morphisms. Moreover, the map $\varphi_+ \circ \iota : H \to A$ takes values in A_+.

Remark 2.8 Proposition 2.3 gives a factorisation in the group (char$(H, A), \star$) of characters. Proposition 2.7 does not encode a group factorisation, because the product $\star$ is derived from a coaction in this case.

Remark 2.9 When $Q = \mathrm{id}_A$, one recovers the recursive definition for the so-called twisted antipode $\tilde{\mathscr{A}} : H \to \hat{H}$:

$$\begin{aligned} \tilde{\mathscr{A}} &= -\iota - m_{\hat{H}}(\mathrm{id}_{\hat{H}} \otimes \tilde{\mathscr{A}})\hat{\Delta}' \circ \iota \\ \varphi_- &= \mathbf{1}^* - \bar{\varphi} \circ \iota = \mathbf{1}^* - m_A((\varphi - \mathbf{1}^*) \otimes \varphi_-)\hat{\Delta} \circ \iota = \varphi \circ \tilde{\mathscr{A}}. \end{aligned}$$

Remark 2.10 Another example of linear maps Q that satisfy (2.3) are idempotent algebra morphisms:

$$Q(f \cdot_A g) = Q(f) \cdot_A Q(g), \quad Q \circ Q = Q.$$

They appear in the negative renormalisation for Regularity Structures [BHZ19], but also in numerical analysis when one wants to perform local error analysis. We refer the reader to [BS20] for details.

Remark 2.11 The map Q can be replaced by a Rota–Baxter family of maps $(Q_\alpha)_{\alpha \in \mathbf{R}_+}$ satisfying the identity

$$(Q_\alpha f) \cdot_A (Q_\beta g) = Q_{\alpha+\beta}(Q_\alpha(f) \cdot_A g + f \cdot_A Q_\beta(g) - f \cdot_A g).$$

In fact, we suppose given a map $|\cdot| : H \to \mathbf{R}_+$ and we will consider the counter term map $\varphi_- = -Q_{|\cdot|} \circ \bar{\varphi} \circ \iota$:

$$\varphi_-(\tau) = -Q_{|\tau|}(\bar{\varphi}(\iota(\tau))).$$

Remark 2.12 In Sections 4, we consider three algebraic Birkhoff type factorisations based on Proposition 2.7. In each case, we will specify the coaction and the target space A respectively A_- and A_+, which are different depending on the application.

- The recentering character for Regularity Structures first introduced in [Hai14] and its construction using a twisted antipode has been made precise in [BHZ19]. The main challenge is that one has to work with a Hopf algebra which is not connected. Therefore, the construction of the twisted antipode and the reduced coproduct map are more involved.
- The negative renormalisation for SPDEs which is reproducing the BPHZ algorithm by incorporating all the Taylor expansions at the level of the algebra. It was introduced in [BHZ19] and is also described by a twisted antipode. This renormalisation is described by Proposition 2.7.
- The character approximating iterated integrals introduced in [BS20]. It is used to provide a numerical scheme for a large class of dispersive PDEs. At first sight, this character appears to be close to the recentering character. However, one can use some simplifications in order to apply directly Proposition 2.7.

3 Bogoliubov-type recursions on regularity structure trees

3.1 Decorated trees

Recall that $\mathcal{RT}$ referred to the set of non-planar rooted trees. Let $\mathfrak{L}$ be a finite set sontaining so-called types. For a given $d \in \mathbf{N}$ we define the set of decorations $\mathfrak{D} = \mathfrak{L} \times \mathbf{N}^{d+1}$ and consider the set $\mathcal{RT}^{\mathfrak{D}}$ of $\mathfrak{D}$-decorated rooted. Elements of $\mathcal{RT}^{\mathfrak{D}}$ are of the form $T^{\mathfrak{n}}_{\mathfrak{e}} = (T, \mathfrak{n}, \mathfrak{e})$ where T is a non-planar rooted tree with node and edge sets N_T respectively E_T. The maps $\mathfrak{n} : N_T \to \mathbf{N}^{d+1}$ and $\mathfrak{e} = (\mathfrak{e}_1, \mathfrak{e}_2) : E_T \to \mathfrak{D}$ are node respectively edge decorations. The tree product $\cdot$ on $\mathcal{RT}^{\mathfrak{D}}$ is defined by

$$(T, \mathfrak{n}, \mathfrak{e}) \cdot (\bar{T}, \bar{\mathfrak{n}}, \bar{\mathfrak{e}}) = (T \cdot \bar{T}, \bar{\mathfrak{n}} + \mathfrak{n}, \bar{\mathfrak{e}} + \mathfrak{e}) , \tag{3.1}$$

where $T \cdot \bar{T}$ is the rooted tree obtained by identifying the roots, the sums $\bar{\mathfrak{n}} + \mathfrak{n}$ and $\bar{\mathfrak{e}} + \mathfrak{e}$ mean that decorations are added at the root and extended to the disjoint union by setting them to vanish on the other tree. In this paper, we will use mainly a symbolic notation for these decorated trees.

1. An edge decorated by $(\mathfrak{t}, p) \in \mathfrak{D}$ is denoted by $\mathcal{I}_{(\mathfrak{t},p)}$. The symbol $\mathcal{I}_{(\mathfrak{t},p)}$ is also viewed as the operation that grafts a tree onto a new root via a new edge with edge decoration $(\mathfrak{t}, p)$
2. A factor X^k encodes a single node $\bullet^k$ decorated by $k \in \mathbf{N}^{d+1}$. We write X_i, $i \in \{1, \ldots, d+1\}$, to denote X^{e_i}, where the $\{e_1, \ldots, e_{d+1}\}$ form the canonical basis of $\mathbf{N}^{d+1}$. The element X^0 is denoted by $\mathbf{1}$.
3. In the following we will employ a drastically simplified notation for decorated trees $\hat{\tau} = T^{\mathfrak{n}}_{\mathfrak{e}} \in \mathcal{RT}^{\mathfrak{D}}$.

Recall from Remark 2.1 the B_+-operation for rooted trees in $\mathcal{RT}$. We employ an analog representation for trees $\hat{\tau} \in \mathcal{RT}^{\mathfrak{D}}$ using $\mathcal{I}_{(\mathfrak{t},p)}$

$$\hat{\tau} = X^{k_0} \prod_{i=1}^{n} \mathcal{I}_{(\mathfrak{t}_i,p_i)}(\hat{\tau}_i),$$

where the $\widehat{\tau}_i$ belong to $\mathcal{RT}^{\mathfrak{D}}$ and the product $\prod_i^n$ is the tree product. The main difference with the B_+-operation is that the edges connecting the new root carry different decorations. Using symbolic notation, one can reformulate the tree product (3.1) as

$$\left(X^{k_0}\prod_i \mathcal{I}_{(\mathfrak{t}_i,p_i)}(\widehat{\tau}_i)\right)\left(X^{k_0'}\prod_j \mathcal{I}_{(\mathfrak{t}_j',p_j')}(\widehat{\tau}_j')\right) = X^{k_0+k_0'}\prod_{i,j}\mathcal{I}_{(\mathfrak{t}_i,p_i)}(\tau_i)\mathcal{I}_{(\mathfrak{t}_j',p_j')}(\widehat{\tau}_j').$$

The space of $\mathfrak{D}$-decorated trees is denoted $\mathcal{T}^{\mathfrak{D}} = \langle \mathcal{RT}^{\mathfrak{D}}\rangle$. Endowed with the tree product it becomes a commutative algebra. We now associate numbers to decorated trees, depending on the decorations. Further below, it will become clear that they have a transparent interpretation in the context of stochastic partial differential equations (SPDEs). Let us fix a scaling $\mathfrak{s} \in \mathbf{N}^{d+1}$ and the associate map $|\cdot|_{\mathfrak{s}} : \mathfrak{L} \to \mathbf{R}$. We extend the latter to $k \in \mathbf{N}^{d+1}$ by $|k|_{\mathfrak{s}} := \sum_{i=1}^{d+1}\mathfrak{s}_i k_i$. The degree of a decorated tree $T_{\mathfrak{e}}^{\mathfrak{n}}$ is defined by

$$|T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} = |\sum_{v\in N_T}\mathfrak{n}(v) + \sum_{e\in E_T}(\mathfrak{e}_1(e)-\mathfrak{e}_2(e))|_{\mathfrak{s}}$$

where $\mathfrak{e} = (\mathfrak{e}_1,\mathfrak{e}_2)$. We further assume that $|\cdot|_{\mathfrak{s}}$ takes values in $\mathbf{R}\setminus\mathbf{Z}$ except for the monomials X^k. Using this degree, we define the set $\mathcal{RT}_+^{\mathfrak{D}}$ which is included in $\mathcal{RT}^{\mathfrak{D}}$ by

$$\mathcal{RT}_+^{\mathfrak{D}} := \left\{X^{k_0}\prod_{i=1}^n \mathcal{I}_{(\mathfrak{t}_i,p_i)}(\widehat{\tau}_i),\, |\mathcal{I}_{(\mathfrak{t}_i,p_i)}(\widehat{\tau}_i)|_{\mathfrak{s}} > 0, \widehat{\tau}_i \in \mathcal{RT}^{\mathfrak{D}}\right\}. \tag{3.2}$$

This definition means that all the branches outgoing from the root must be of positive degree. We denote by $\mathcal{T}_+^{\mathfrak{D}}$ the space $\langle\mathcal{RT}_+^{\mathfrak{D}}\rangle$ and call it the positive part. The corresponding projector π_+ maps $\mathcal{T}^{\mathfrak{D}}$ to $\mathcal{T}_+^{\mathfrak{D}}$. In the following we denote by $\mathcal{M}$ the tree product on $\mathcal{T}^{\mathfrak{D}}$.

Remark 3.1 In some examples, one wants to restrict the tree product to a subspace of $\mathcal{T} \subset \mathcal{T}^{\mathfrak{D}}$ which is not an algebra. However, we would like to keep the algebra structure on the positive part $\mathcal{T}_+$ of $\mathcal{T}$, which is not included in $\mathcal{T}$. The definition of $\mathcal{T}_+$ is similar as $\mathcal{RT}_+^{\mathfrak{D}}$. With the symbolic notation, the difference is made at the root where the edges outgoing the root for a tree in $\mathcal{T}_+^{\mathfrak{D}}$ are denoted by $\hat{\mathcal{J}}$. This means that we put a colour at the root or that the root has been distinguished see Section 4.1.

3.2 Comodule-Hopf algebra structures

We want to endow the previously introduced algebra on decorated trees with a coproduct which is similar to the Butcher–Connes–Kreimer coproduct [CK98] on rooted trees (2.1). However, primitiveness of many elements is lost due to the particular nature of the decorations of trees in our setting. We will provide a recursive definition of the coproduct Δ^+ which is similar to (2.1) and suffices for formulating the main result, i.e., an algebraic Birkhoff-type factorisation. The map Δ^+ is recursively defined on $\mathcal{T}^{\mathfrak{D}}$:

$$\Delta^+\mathbf{1} = \mathbf{1}\otimes\mathbf{1}, \quad \Delta^+ X_i = X_i\otimes\mathbf{1} + \mathbf{1}\otimes X_i,$$

$$\Delta^+\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau}) = \mathbf{1}\otimes\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau}) + (\mathcal{I}_{(\mathfrak{t},p)}\otimes \mathrm{id})\Delta^+\widehat{\tau} + \sum_{\substack{\ell\in\mathbf{N}^{d+1}\\ \ell\neq 0}} \frac{X^\ell}{\ell!}\otimes\mathcal{I}_{(\mathfrak{t},p+\ell)}(\widehat{\tau}). \tag{3.3}$$

Recall that $\mathbf{1}^* : \mathcal{T}^{\mathfrak{D}} \to \mathbf{R}$ refers to counit. The map Δ^+ is given on various domains and for clarity we will denote it differently:

- $\Delta^+ : \mathcal{T}^{\mathfrak{D}} \to \mathcal{T}^{\mathfrak{D}}\otimes\mathcal{T}^{\mathfrak{D}}$. Here, a specific bigrading is required. See [BHZ19, Section 2.3] for more details. A possible choice would be
$$(|T^{\mathfrak{n}}_{\mathfrak{e}}|_{bi}) = (|\mathfrak{e}|_{\mathfrak{s}}, |N_T\setminus\{\varrho_T\}| + |E_T|),$$
where $|\mathfrak{e}|_{\mathfrak{s}} = \sum_{e\in E_T}|\mathfrak{e}(e)|_{\mathfrak{s}}$, ϱ_T is the root vertex of T, $|N_T|$ and $|E_T|$ are the numbers of nodes and edges for the tree T. This map will have the interpretation of performing an infinite subtraction. Its recursive description appears in [BHZ19, Proposition 4.16].
- $\hat{\Delta}^+ = (\mathrm{id}\otimes\pi_+)\Delta^+ : \mathcal{T}^{\mathfrak{D}} \to \mathcal{T}^{\mathfrak{D}}\otimes\mathcal{T}^{\mathfrak{D}}_+$. Here, no bigrading is required, as the sum in (3.3) is finite. It should be understood as finite subtraction, with its length determined by the degree of the branch outgoing from the root vertex.
- $\bar{\Delta}^+ = (\pi_+\otimes\pi_+)\Delta^+ : \mathcal{T}^{\mathfrak{D}}_+ \to \mathcal{T}^{\mathfrak{D}}_+\otimes\mathcal{T}^{\mathfrak{D}}_+$, as before no bigrading is required. We put an extra assumption on the trunk by maintaining the positive degree of the branches outgoing from the root. This is a rather strong constraint because the degree of the trunk is lower than that of the original tree (branches of positive degree have been removed).

Remark 3.2 In [BS20], a similar coproduct as (3.3) is used. The main difference relies on the projection π_+. Indeed, for a numerical scheme the length of the Taylor expansion depends on the order of the scheme, whereas in our context it depends on the regularity of the distribution we would like to re-center.

Following [BHZ19, Proposition 3.23], we are able to put a Hopf algebra structure on the different sets of decorated trees:

Proposition 3.3 *We have the following properties*

- *Considering the space $\mathcal{T}^{\mathfrak{D}}$ of decorated trees, there exists an algebra morphism $\mathcal{A}_+ : \mathcal{T}^{\mathfrak{D}} \to \mathcal{T}^{\mathfrak{D}}$ such that $H^{\mathfrak{D}} = (\mathcal{T}^{\mathfrak{D}}, \mathcal{M}, \Delta^+, \mathbf{1}, \mathbf{1}^*, \mathcal{A}_+)$ is a Hopf algebra.*
- *Considering the positive part $\mathcal{T}^{\mathfrak{D}}_+ \subset \mathcal{T}^{\mathfrak{D}}$, there exists an algebra morphism $\bar{\mathcal{A}}_+ : \mathcal{T}^{\mathfrak{D}}_+ \to \mathcal{T}^{\mathfrak{D}}_+$ so that $H^{\mathfrak{D}}_+ = (\mathcal{T}^{\mathfrak{D}}_+, \mathcal{M}, \bar{\Delta}^+, \mathbf{1}, \mathbf{1}^*, \bar{\mathcal{A}}_+)$ is a Hopf algebra.*
- *The map $\hat{\Delta}^+ : \mathcal{T}^{\mathfrak{D}} \to \mathcal{T}^{\mathfrak{D}}\otimes\mathcal{T}^{\mathfrak{D}}_+$ is a coaction satisfying:*
$$\left(\hat{\Delta}^+\otimes\mathrm{id}\right)\hat{\Delta}^+ = \left(\mathrm{id}\otimes\bar{\Delta}^+\right)\hat{\Delta}^+$$

and turns $\mathcal{T}^{\mathfrak{D}}$ into a right-comodule for $\mathcal{T}_+^{\mathfrak{D}}$.

Remark 3.4 The antipode for $H^{\mathfrak{D}}$ is described in terms of a recursive formula given in [BHZ19, Proposition 4.18] by

$$\begin{aligned}\mathcal{A}_+ X_i &= -X_i,\\ \mathcal{A}_+ \mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau}) &= -\sum_{\ell\in\mathbf{N}^{d+1}} \frac{(-X)^\ell}{\ell!}\mathcal{M}(\mathcal{I}_{(\mathfrak{t},p+\ell)}\otimes\mathcal{A}_+)\Delta^+\widehat{\tau}.\end{aligned}$$

The infinite sum is made rigorous by use of a specific bigrading. For the antipode of $H_+^{\mathfrak{D}}$, we get a similar definition [BHZ19, Proposition 6.2]:

$$\begin{aligned}\bar{\mathcal{A}}_+ X_i &= -X_i,\\ \bar{\mathcal{A}}_+ \mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau}) &= -\sum_{\ell\in\mathbf{N}^{d+1}} \frac{(-X)^\ell}{\ell!}\mathcal{M}\big(\pi_+\mathcal{I}_{(\mathfrak{t},p+\ell)}\otimes\bar{\mathcal{A}}_+\big)\hat{\Delta}^+\widehat{\tau}.\end{aligned} \tag{3.4}$$

However, the difference is the introduction of the projector π_+ as well as the coaction $\hat{\Delta}^+$, which make the sum finite.

In fact, the map which is of interest to us, is the one where the projector π_+ is not used in (3.4). This map will be denoted $\tilde{\mathcal{A}}_+ : \mathcal{T}_+^{\mathfrak{D}} \to \mathcal{T}^{\mathfrak{D}}$ and is given in [BHZ19, Proposition 6.3] by:

$$\begin{aligned}\tilde{\mathcal{A}}_+ X_i &= -X_i,\\ \tilde{\mathcal{A}}_+ \mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau}) &= -\sum_{|\ell|_{\mathfrak{s}}\leq|\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau})|_{\mathfrak{s}}} \frac{(-X)^\ell}{\ell!}\mathcal{M}\big(\mathcal{I}_{(\mathfrak{t},p+\ell)}\otimes\tilde{\mathcal{A}}_+\big)\hat{\Delta}^+\widehat{\tau}.\end{aligned} \tag{3.5}$$

It is called twisted antipode and plays a major role in describing the local behaviour of solutions of singular SPDEs. In (3.5), the projection π_+ is replaced by a global one which is based on the degree of the decorated tree on which we apply the twisted antipode. In this work we will reinterpret this map as a Bogoliubov-type recursion. The aim is to bridge the gap between the renormalisation procedure developed for singular SPDEs and Connes–Kreimer's formulation of the BPHZ renormalisation procedure in perturbative QFT in terms of an algebraic Birkhoff factorisation on the level of regularised Feynman rules seen as a Hopf algebra character. In the next section, we introduce a modified reduced coproduct together with a Rota–Baxter-type map essential for Bogoliubov-type recursions.

Before stating the definition of a modified reduced coproduct, we notice that the decorated tree $\bullet^n = X^n$ is not primitive with respect to the coproduct (3.3). Indeed

$$\Delta^+ X^n = X^n \otimes \mathbf{1} + \mathbf{1}\otimes X^n + \sum_{\substack{k\in\mathbf{N}^{d+1}\\ k\neq 0,n}} \binom{n}{k} X^k \otimes X^{n-k},$$

where the binomial coefficient $\binom{n}{k} := \prod_{i=1}^{d+1} \binom{n_i}{k_i}$ and $\binom{n_i}{k_i}$ is zero when k_i is bigger than n_i. We refer the reader to [BHZ19, Section 2] for details. For a tree of the form $\mathcal{I}_{(\mathfrak{t},p+\ell)}(\hat{\tau})$, called planted tree, the main difference with the Butcher–Connes–Kreimer coproduct (**??**) is that one goes to the next orders by adding derivatives and polynomial decorations in (3.3). These extra terms are given by:

$$\sum_{\substack{\ell \in \mathbf{N}^{d+1} \\ \ell \neq 0}} \frac{X^\ell}{\ell!} \otimes \mathcal{I}_{(\mathfrak{t},p+\ell)}(\hat{\tau}).$$

A natural choice is to also remove those terms from a potential definition of what we will call modified reduced coproduct. We need to generalise this procedure to arbitrary trees, i.e., products of planted trees. The basic idea is to remove any polynomial part on the righthand side of the modified reduced coproduct.

Definition 3.5 The modified reduced coproduct map Δ^+_{red} is given on X^{k_0} by:

$$\Delta^+_{\text{red}} X^{k_0} = 0.$$

Then for any rooted tree $\hat{\tau} = X^{k_0} \prod_{i=1}^n \mathcal{I}_{(\mathfrak{t}_i,p_i)}(\hat{\tau}_i) \in \mathcal{T}^{\mathfrak{D}}$ we set:

$$\begin{aligned} \Delta^+_{\text{red}} \hat{\tau} &= \hat{\Delta}^+ \hat{\tau} - \hat{\tau} \otimes \mathbf{1} \\ &\quad - \sum_{\ell_i, k \in \mathbf{N}^{d+1}} \frac{1}{\bar{\ell}!} \binom{k_0}{k} X^{k + \sum_i \ell_i} \otimes X^{k_0 - k} \prod_{i=1}^n \pi_+ \mathcal{I}_{(\mathfrak{t}_i, p_i + \ell_i)}(\hat{\tau}_i) \end{aligned} \tag{3.6}$$

where $\bar{\ell}! = \prod_i \ell_i!$ and for $k_0 = 0$, the sum over k contains only the term $k = 0$ by convention.

Remark 3.6 Sweedler's notation is used for the modified reduced coproduct (3.6):

$$\Delta^+_{\text{red}} \hat{\tau} = \sum_{(\hat{\tau})}^{+} \hat{\tau}' \otimes \hat{\tau}''.$$

Remark 3.7 The definition of the modified reduced coproduct implies the primitiveness of $\bullet^n$, i.e., $\Delta^+_{\text{red}} X^n = 0$. Moreover, one gets the following recursion:

$$\Delta^+_{\text{red}} \mathcal{I}_{(\mathfrak{t},p)}(\hat{\tau}) = (\mathcal{I}_{(\mathfrak{t},p)} \otimes \gamma) \hat{\Delta}^+ \hat{\tau}, \tag{3.7}$$

where $\gamma = \text{id} - \mathbf{1}^*$ is the augmentation projector, which is zero on the empty tree, $\mathbf{1}$, and the identity else.

3.3 Bogoliubov's recursion

After having introduced the modified reduced coproduct (3.6), we need to consider the space of characters, which will be used to iterate the Bogoliubov-type recursion. They are linear maps from $H^{\mathfrak{D}}$ to the specific target space of functions, $\mathcal{H} =$

$\mathscr{C}^\infty(\mathbf{R}^{d+1}, \mathbf{R})$, respecting the tree product $\mathscr{M}$. The recursion we want to set up will produce a decomposition of a specific character into a product two characters, and will depend on a parameter $x \in \mathbf{R}^{d+1}$. The latter is used to fix the corresponding splitting of the target space

$$\mathscr{H} = \mathscr{H}_x^+ \oplus \mathscr{H}_x^-,$$

where $\mathscr{H}_x^+$ contains the functions vanishing at x and $\mathscr{H}_x^-$ consists of polynomial functions whose coefficients are functions of x. Indeed, for any $f \in \mathscr{H}$ one has the straightforward splitting $f = f - f(x) + f(x)$. The next definition presents the key map lying at the origin of the Bogoliubov-type recursion. It has to be understood as a Taylor jet (of order α).

Definition 3.8 We set for $\alpha \in \mathbf{R}_+$, $x, y \in \mathbf{R}^{d+1}$ and $f \in \mathscr{H}$

$$\mathrm{T}_{\alpha,x,y} f \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \mathbf{N}^{d+1} \\ |\ell|_{\mathfrak{s}} < \alpha}} \frac{(y-x)^\ell}{\ell!} (D^\ell f)(x). \tag{3.8}$$

The next lemma provides some key properties of the Taylor jet (3.8) which will be used in the sequel.

Lemma 3.9 *The operators* $\mathrm{T}_{\cdot,x,\cdot}$ *defined in* (3.8) *satisfy for every* $\alpha, \beta \in \mathbf{R}_+$ *and functions* $f, g \in \mathscr{C}^\infty(\mathbf{R}^{d+1}, \mathbf{R})$ *the following Rota–Baxter-type identity*

$$(\mathrm{T}_{\alpha,x,\cdot} f)(\mathrm{T}_{\beta,x,\cdot} g) = \mathrm{T}_{\alpha+\beta,x,\cdot}[(\mathrm{T}_{\alpha,x,\cdot} f) g + f (\mathrm{T}_{\beta,x,\cdot} g) - fg]. \tag{3.9}$$

For a fixed but arbitrary $\bar{x} \in \mathbf{R}^{d+1}$ *we have*

$$(\mathrm{T}_{\alpha,x,y} f) = \sum_{|\ell|_{\mathfrak{s}} < \alpha} \frac{(y-\bar{x})^\ell}{\ell!} \mathrm{T}_{\alpha-|\ell|_{\mathfrak{s}},x,\bar{x}}[D^\ell f]. \tag{3.10}$$

Proof. The first identity (3.9) is well-known in the literature and corresponds to the notion of family of Rota–Baxter maps [EGBP07]. We give a proof of the second one (3.10) which is in fact essential for the sequel. One has

$$\begin{aligned}
\sum_{|\ell|_{\mathfrak{s}} < \alpha} \frac{(y-\bar{x})^\ell}{\ell!} \mathrm{T}_{\alpha-|\ell|_{\mathfrak{s}},x,\bar{x}}[D^\ell f] &= \sum_{|\ell|_{\mathfrak{s}} < \alpha} \frac{(y-\bar{x})^\ell}{\ell!} \sum_{|k|_{\mathfrak{s}} < \alpha - |\ell|_{\mathfrak{s}}} \frac{(\bar{x}-x)^k}{k!} D^{\ell+k} f(x) \\
&= \sum_{|\ell|_{\mathfrak{s}} < \alpha} \frac{(y-\bar{x}+\bar{x}-x)^\ell}{\ell!} D^\ell f(x) \\
&= \mathrm{T}_{\alpha,x,y} f
\end{aligned}$$

□

Remark 3.10 Identity (3.9) is also true for $\alpha, \beta \in \mathbf{R}_-$ by setting $\mathrm{T}_{\alpha,x,\cdot} f = 0$ whenever $\alpha \leq 0$.

We consider now linear maps from $H^{\mathscr{D}}$ to $\mathscr{H}$ parametrised by finite sets of elements in $\mathbf{R}^{d+1}$. Of particular interest is the family of algebra morphisms

$$\Phi = (\varphi_{x_1,\ldots,x_n})_{x_1,\ldots,x_n \in \mathbf{R}^{d+1}}, \quad \varphi_{x_1,\ldots,x_n} : H^{\mathscr{D}} \to \mathscr{H}.$$

The basic idea is that a priori some origin has been fixed and that characters $\varphi_{\bar{x}} : H^{\mathscr{D}} \to \mathscr{H}$ contain partially a re-centering. Indeed, they are defined on $\bullet^n$ and the natural evaluation should give the polynomial function associated to X^k. Therefore, our polynomial functions will be re-centered around this parameter.

We consider now the linear maps $\varphi_{\bar{x}}\colon H^{\mathscr{D}} \to \mathscr{H}$, $\varphi^-_{x,\bar{x}} : H^{\mathscr{D}}_+ \to \mathscr{H}^-_x$, $\bar{\varphi}_{x,\bar{x}} : H^{\mathscr{D}} \to \mathscr{H}$ and $\varphi^+_{x,\bar{x}} : H^{\mathscr{D}} \to \mathscr{H}$ in Φ and follow Proposition 2.7 in setting up a Bogoliubov-type recursion for the counter term map. It is assumed that $\varphi_{\bar{x}}$ is a character parametrised by $\bar{x} \in \mathbf{R}^{d+1}$. The map $\bar{\varphi}_{x,\bar{x}}$ plays the role of Bogoliubov's preparation map obtained from the modified reduced coproduct Δ^+_{red}. Upon applying the Taylor jet operator Bogoliubov's preparation map gives the counter term character, $\varphi^-_{x,\bar{x}}$. Eventually, the renormalised character $\varphi^+_{x,\bar{x}}$ follows from a convolution between $\varphi^-_{x,\bar{x}}$ and $\varphi^-_{\bar{x}}$ using the coaction $\hat{\Delta}^+$. We will skip the injection from $H^{\mathscr{D}}_+$ into $H^{\mathscr{D}}$ for notational clarity.

Definition 3.11 Let $\varphi_{\bar{x}} \in \Phi$, $\bar{x} \in \mathbf{R}^{d+1}$ be a character. We set up the following Bogoliubov-type recursion with respect to the points $x, y \in \mathbf{R}^{d+1}$, $\hat{\tau} \in \mathscr{R}\mathscr{T}^{\mathscr{D}}$:

$$\begin{aligned}
\bar{\varphi}_{x,\bar{x},y}(\hat{\tau}) &= \varphi_{\bar{x},y}(\hat{\tau}) + \sum_{(\hat{\tau})}^{+} \varphi_{\bar{x},y}(\hat{\tau}')\varphi^-_{x,\bar{x},\bar{x}}(\hat{\tau}'') \\
\varphi^-_{x,\bar{x},y}(\hat{\tau}) &= -\mathrm{T}_{|\hat{\tau}|_{\mathfrak{s}},x,y}(\bar{\varphi}_{x,\bar{x},\cdot}(\hat{\tau})) \\
\varphi^+_{x,\bar{x}} &= \varphi_{\bar{x}} \star \varphi^-_{x,\bar{x},\cdot_{|\bar{x}}} = \Big(\varphi_{\bar{x}} \otimes \varphi^-_{x,\bar{x},\cdot_{|\bar{x}}}\Big)\hat{\Delta}^+.
\end{aligned} \tag{3.11}$$

Remark 3.12 Using the recursive formulation of the modified reduced coproduct (3.7), one gets the following identity for $\bar{\varphi}_{x,\bar{x},\cdot}$

$$\bar{\varphi}_{x,\bar{x},y}(\mathcal{I}_{(\mathfrak{t},p)}(\hat{\tau})) = \varphi_{\bar{x},y}(\mathcal{I}_{(\mathfrak{t},p)}(\hat{\tau})) + \Big(\varphi_{\bar{x},\cdot_{|y}}\mathcal{I}_{(\mathfrak{t},p)} \otimes \varphi^-_{x,\bar{x},\cdot_{|\bar{x}}}\Big)\hat{\Delta}^+\hat{\tau}. \tag{3.12}$$

The central point is to verify the following properties:

1. The counter term map $\varphi^-_{x,\bar{x}}$ is an algebra morphism from $H^{\mathscr{D}}_+$ into $\mathscr{H}^-_x$.
2. The renormalised map $\varphi^+_{x,\bar{x}}$ is an algebra morphism from $H^{\mathscr{D}}$ into $\mathscr{H}$, which sends trees from $\mathscr{T}^{\mathscr{D}}_+$ into $\mathscr{H}^+_x$.
3. One can prove various $\bar{x}$-invariances for the maps $\varphi^+_{x,\bar{x}}$, $\varphi^-_{x,\bar{x}}$ and $\bar{\varphi}_{x,\bar{x}}$.

Assumption 1 *We assume that the family of characters $(\varphi_{\bar{x}})_{\bar{x} \in \mathbf{R}^{d+1}}$ satisfies:*

$$\varphi_{\bar{x},y}(X_i) = y_i - \bar{x}_i, \quad \varphi_{\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},k+\ell)}(\hat{\tau})) = D^{\ell}\varphi_{\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},k)}(\hat{\tau})).$$

The first identity in Assumption 1 corresponds to interpreting the point $\bar{x}$ as a re-centering of polynomials. The second identity shows that adding decoration to an edge amounts in fact to taking derivatives. This identity is crucial and combined with (3.10) allows us to prove the character property of the counter term $\varphi^-_{x,\bar{x}}$.

From (3.12) and Assumption 1, we have for $\mathcal{I}_{(\mathfrak{t},k+\ell)}(\widehat{\tau}) \in \mathcal{T}_+^{\mathcal{D}}$ that

$$\bar{\varphi}_{x,\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},k+\ell)}(\widehat{\tau})) = D^\ell \bar{\varphi}_{x,\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},k)}(\widehat{\tau})). \tag{3.13}$$

Assumption 1 gives also the behaviour of Bogoliubov's recursion on the polynomials. From the primitiveness, $\Delta^+_{\text{red}} X^k = 0$, one gets:

$$\bar{\varphi}_{x,\bar{x},y}(X^k) = \varphi_{\bar{x},y}(X^k) = (y-x)^k.$$

Then,

$$\begin{aligned}\varphi^-_{x,\bar{x},y}(X^k) &= -\mathrm{T}_{|k|_\mathfrak{s},x,y}\Big(\bar{\varphi}_{x,\bar{x},\cdot}(X^k)\Big) = -\mathrm{T}_{|k|_\mathfrak{s},x,y}\Big((\cdot-\bar{x})^k\Big) \\ &= -\sum_{|\ell|_\mathfrak{s}<|k|_\mathfrak{s}} \binom{k}{\ell}(y-x)^\ell (x-\bar{x})^{k-\ell}.\end{aligned}$$

For $y=\bar{x}$, we then have

$$\begin{aligned}\varphi^-_{x,\bar{x},\bar{x}}(X^k) &= -\sum_{|\ell|_\mathfrak{s}<|k|_\mathfrak{s}} \binom{k}{\ell}(\bar{x}-x)^\ell (x-\bar{x})^{k-\ell} \\ &= -\sum_{|\ell|_\mathfrak{s}\leq|k|_\mathfrak{s}} \binom{k}{\ell}(\bar{x}-x)^\ell (x-\bar{x})^{k-\ell} + (\bar{x}-x)^k \\ &= (\bar{x}-x)^k.\end{aligned}$$

At the end,

$$\begin{aligned}\varphi^+_{x,\bar{x},y}(X^k) &= \big(\varphi_{\bar{x},y}\otimes\varphi^-_{x,\bar{x},\bar{x}}\big)\hat{\Delta}^+ X^k \\ &= \sum_{|\ell|_\mathfrak{s}\leq|k|_\mathfrak{s}} \binom{k}{\ell}\varphi_{\bar{x},y}(X^\ell)\varphi^-_{x,\bar{x},\bar{x}}(X^{k-\ell}) \\ &= \sum_{|\ell|_\mathfrak{s}\leq|k|_\mathfrak{s}} \binom{k}{\ell}(y-\bar{x})^\ell(\bar{x}-x)^{k-\ell} \\ &= (y-x)^k.\end{aligned}$$

Theorem 3.13 *The map $\varphi^-_{x,\bar{x}}$ is an algebra morphism from $H_+^{\mathcal{D}}$ into $\mathcal{H}_x^-$ and $\varphi^+_{x,\bar{x}}$ is an algebra morphism from $H^{\mathcal{D}}$ into $\mathcal{H}$.*

Proof. We proceed by induction on the number of edges and the decoration at the root. We consider trees $\widehat{\tau}_1, \widehat{\tau}_2 \in \mathcal{R}\mathcal{T}_+^{\mathcal{D}}$. According to our convention they both admit symbolic representations of the form:

$$\widehat{\tau}_1 = X^n \prod_i \mathcal{I}_{(\mathfrak{t}_i,k_i)}(\widehat{\tau}_{1i}), \quad \widehat{\tau}_2 = X^{\bar{n}} \prod_j \mathcal{I}_{(\bar{\mathfrak{t}}_j,\bar{k}_j)}(\widehat{\tau}_{2j}).$$

We expand the recursive expression of the counter term $\varphi^-_{x,\bar{x}}$ giving for $y \in \mathbf{R}^{d+1}$:

$$
\begin{aligned}
\varphi^-_{x,\bar{x},y}(\widehat{\tau}_1\widehat{\tau}_2) &= -\mathrm{T}_{|\widehat{\tau}_1\widehat{\tau}_2|_\mathfrak{s},x,y}\left(\varphi_{\bar{x},\cdot}(\widehat{\tau}_1\widehat{\tau}_2) + \sum_{(\widehat{\tau}_1\widehat{\tau}_2)}^{+} \varphi_{\bar{x},\cdot}\big(\{\widehat{\tau}_1\widehat{\tau}_2\}'\big)\varphi^-_{x,\bar{x},\bar{x}}\big(\{\widehat{\tau}_1\widehat{\tau}_2\}''\big)\right) \\
&= -\mathrm{T}_{|\widehat{\tau}_1\widehat{\tau}_2|_\mathfrak{s},x,y}\left(\left\{\varphi_{\bar{x},\cdot}(\widehat{\tau}_1) + \sum_{(\widehat{\tau}_1)}^{+} \varphi_{\bar{x},\cdot}(\widehat{\tau}_1')\varphi^-_{x,\bar{x},\bar{x}}\big(\widehat{\tau}_1''\big)\right\}\right. \\
&\left\{\varphi_{\bar{x},\cdot}(\widehat{\tau}_2) + \sum_{(\widehat{\tau}_2)}^{+} \varphi_{\bar{x},\cdot}(\widehat{\tau}_2')\varphi^-_{x,\bar{x},\bar{x}}\big(\widehat{\tau}_2''\big)\right\} \\
&+ \sum_{\substack{\ell_1,\dots,\ell_{d+1} \\ |k'|_\mathfrak{s} < |n|_\mathfrak{s}}} \binom{n}{k'}\varphi_{\bar{x},\cdot}\left(\frac{X^{\sum_i \ell_i + k'}}{\bar{\ell}!}\right)\varphi^-_{x,\bar{x},\bar{x}}\left(X^{n-k'}\prod_j \mathcal{I}_{(\mathfrak{t}_j,k_j+\ell_j)}(\widehat{\tau}_{1j})\right) \\
&\left\{\varphi_{\bar{x},\cdot}(\widehat{\tau}_2) + \sum_{(\widehat{\tau}_2)}^{+} \varphi_{\bar{x},\cdot}\big(\widehat{\tau}_2'\big)\varphi^-_{x,\bar{x},\bar{x}}\big(\widehat{\tau}_2''\big)\right\} \\
&+ \sum_{\substack{\ell_1,\dots,\ell_{d+1} \\ |k'|_\mathfrak{s} < |\bar{n}|_\mathfrak{s}}} \binom{\bar{n}}{k'}\varphi_{\bar{x},\cdot}\left(\frac{X^{\sum_i \ell_i + k'}}{\bar{\ell}!}\right)\varphi^-_{x,\bar{x},\bar{x}}\left(X^{\bar{n}-k'}\prod_j \mathcal{I}_{(\bar{\mathfrak{t}}_j,\bar{k}_j+\ell_j)}(\widehat{\tau}_{2j})\right) \\
&\left.\left\{\varphi_{\bar{x},\cdot}(\widehat{\tau}_1) + \sum_{(\widehat{\tau}_1)}^{+} \varphi_{\bar{x},\cdot}\big(\widehat{\tau}_1'\big)\varphi^-_{x,\bar{x},\bar{x}}\big(\widehat{\tau}_1''\big)\right\}\right),
\end{aligned}
$$

where $\bar{\ell}! = \prod_i \ell_i!$. By applying the induction hypothesis on each $X^{n-k}\prod_i \mathcal{I}_{(\mathfrak{t}_i,k_i+\ell_i)}(\widehat{\tau}_{1i})$, we obtain:

$$
\varphi^-_{x,\bar{x},\bar{x}}\left(X^{n-k}\prod_j \mathcal{I}_{(\mathfrak{t}_j,k_j+\ell_j)}(\widehat{\tau}_{1j})\right) = \varphi^-_{x,\bar{x},\bar{x}}\Big(X^{n-k}\Big)\prod_j \varphi^-_{x,\bar{x},\bar{x}}\big(\mathcal{I}_{(\mathfrak{t}_j,k_j+\ell_j)}(\widehat{\tau}_{1j})\big).
$$

We use identity (3.10) to get for each $\mathcal{I}_{(\mathfrak{t}_i,k_i)}(\widehat{\tau}_{1i})$ with $\alpha_i := |\mathcal{I}_{(\mathfrak{t}_i,k_i)}(\widehat{\tau}_{1i})|_\mathfrak{s}$:

$$
\begin{aligned}
\varphi^-_{x,\bar{x},y}\big(\mathcal{I}_{(\mathfrak{t}_i,k_i)}(\widehat{\tau}_{1i})\big) &= -\mathrm{T}_{\alpha_i,x,y}\big(\bar{\varphi}_{x,\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t}_i,k_i)}(\widehat{\tau}_{1i}))\big) \\
&= -\sum_{|\ell_j|_\mathfrak{s} < \alpha_i} \frac{(y-\bar{x})^{\ell_j}}{\ell_j!}\mathrm{T}_{\alpha_i-\ell_j,x,\bar{x}}\big[D^{\ell_j}\bar{\varphi}_{x,\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t}_i,k_i)}(\widehat{\tau}_{1i}))\big] \\
&= -\sum_{|\ell_j|_\mathfrak{s} < \alpha_i} \frac{(y-\bar{x})^{\ell_j}}{\ell_j!}\mathrm{T}_{\alpha_i-\ell_j,x,\bar{x}}\big[\bar{\varphi}_{x,\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t}_i,k_i+\ell_j)}(\widehat{\tau}_{1i}))\big] \\
&= \sum_{|\ell_j|_\mathfrak{s} < \alpha_i} \varphi_{\bar{x},\cdot}\left(\frac{X^{\ell_j}}{\ell_j!}\right)\varphi^-_{x,\bar{x},\bar{x}}\big(\mathcal{I}_{(\mathfrak{t}_i,k_i+\ell_j)}(\widehat{\tau}_{1i})\big).
\end{aligned}
$$

On the other hand we get:

$$\varphi^-_{x,\bar{x},\cdot}(X^n) = \sum_{|k|_{\mathfrak{s}}<|n|_{\mathfrak{s}}} \binom{n}{k} \varphi_{\bar{x},\cdot}\Big(X^k\Big) \varphi^-_{x,\bar{x},\bar{x}}\Big(X^{n-k}\Big).$$

We conclude by applying identity (3.9)

$$\begin{aligned}\varphi^-_{x,\bar{x},y}(\widehat{\tau}_1\widehat{\tau}_2) &= -\mathrm{T}_{|\widehat{\tau}_1\widehat{\tau}_2|_{\mathfrak{s}},x,y}\big(\bar{\varphi}_{x,\bar{x},\cdot}(\widehat{\tau}_1)\bar{\varphi}_{x,\bar{x},\cdot}(\widehat{\tau}_2) - \mathrm{T}_{|\widehat{\tau}_1|_{\mathfrak{s}},x,\cdot}(\bar{\varphi}_{x,\bar{x},\cdot}(\widehat{\tau}_1))\bar{\varphi}_{x,\bar{x},\cdot}(\widehat{\tau}_2) \\ &\quad - \bar{\varphi}_{x,\bar{x},\cdot}(\widehat{\tau}_1)\mathrm{T}_{|\widehat{\tau}_2|_{\mathfrak{s}},x,\cdot}(\bar{\varphi}_{x,\bar{x},\cdot}(\widehat{\tau}_2))\big) \\ &= \mathrm{T}_{|\widehat{\tau}_1|_{\mathfrak{s}},x,y}(\bar{\varphi}_{x,\bar{x},\cdot}(\widehat{\tau}_1))\mathrm{T}_{|\widehat{\tau}_2|_{\mathfrak{s}},x,y}(\bar{\varphi}_{x,\bar{x},\cdot}(\widehat{\tau}_2)) \\ &= \varphi^-_{x,\bar{x},y}(\widehat{\tau}_1)\varphi^-_{x,\bar{x},y}(\widehat{\tau}_2).\end{aligned}$$

The fact that $\varphi^+_{x,\bar{x},y}$ is a character follows from its definition together with $\varphi_{\bar{x}}$ and $\varphi^-_{x,\bar{x},y}$ being characters. □

Remark 3.14 From the proof above, one can see that for a tree $\widehat{\tau}$ with a negative sub-branche we get $\varphi^-_{x,\bar{x},\cdot}(\widehat{\tau}) = 0$ by using the extension mentioned in remark 3.10.

In the next proposition, we connect the twisted antipode, $\tilde{\mathscr{A}}_+ : \mathcal{T}^{\mathfrak{D}}_+ \to \mathcal{T}^{\mathfrak{D}}$, given in (3.5) with the character $\varphi^-_{x,\bar{x},\bar{x}} : H^{\mathfrak{D}}_+ \to \mathbf{R}$ constructed through the Bogoliubov-type recursion.

Proposition 3.15 *Let $\varphi_{\bar{x},x} : H^{\mathfrak{D}} \to \mathbf{R}$ be a character. One has the following identity between characters from $H^{\mathfrak{D}}_+$ to $\mathbf{R}$*

$$\varphi^-_{x,\bar{x},\bar{x}} = \varphi_{\bar{x},x}\tilde{\mathscr{A}}_+.$$

Proof. We proceed by induction on trees. Recall that Assumption 1 is in place. By multiplicativity, we just consider a planted tree $\mathcal{I}_{(\mathfrak{t},k)}(\widehat{\tau})$ and set $\alpha = |\mathcal{I}_{(\mathfrak{t},k)}(\widehat{\tau})|_{\mathfrak{s}}$:

$$\varphi_{\bar{x},x}\big(\tilde{\mathscr{A}}_+\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau})\big) = -\sum_{|\ell|_{\mathfrak{s}}\leq\alpha} \frac{\varphi_{\bar{x},x}(-X)^\ell}{\ell!}\big(\varphi_{\bar{x},x}\mathcal{I}_{(\mathfrak{t},p+\ell)} \otimes \varphi_{\bar{x},x}\tilde{\mathscr{A}}_+\big)\hat{\Delta}^+\widehat{\tau}.$$

We use the inductive hypothesis and $\varphi_{\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},p+\ell)}(\widehat{\tau})) = D^\ell\varphi_{\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau}))$ to get

$$\begin{aligned}\varphi_{\bar{x},x}\big(\tilde{\mathscr{A}}_+\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau})\big) &= -\sum_{|\ell|_{\mathfrak{s}}\leq\alpha}\frac{(\bar{x}-x)^\ell}{\ell!}\Big(\Big(D^\ell\varphi_{\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},p)}(\cdot))\Big)(x)\otimes\varphi^-_{x,\bar{x},\bar{x}}\Big)\hat{\Delta}^+\widehat{\tau} \\ &= -\big(\mathrm{T}_{\alpha,x,\bar{x}}(\varphi_{\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},p)}(\cdot)))\otimes\varphi^-_{x,\bar{x},\bar{x}}\big)\hat{\Delta}^+\widehat{\tau} \\ &= -\mathrm{T}_{\alpha,x,\bar{x}}(\varphi_{\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau}))) - \big(\mathrm{T}_{\alpha,x,\bar{x}}(\varphi_{\bar{x},\cdot})\otimes\varphi^-_{x,\bar{x},\bar{x}}\big)\Delta^+_{\text{red}}\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau}) \\ &= -\mathrm{T}_{\alpha,x,\bar{x}}(\bar{\varphi}_{x,\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau}))) \\ &= \varphi^-_{x,\bar{x},\bar{x}}(\mathcal{I}_{(\mathfrak{t},p)}(\widehat{\tau})).\end{aligned}$$

□

Given a tree $\hat{\tau} = X^n \prod_i \mathcal{I}_{(\mathfrak{t}_i,k_i)}(\hat{\tau}_i)$, with $\alpha_i = |\mathcal{I}_{(\mathfrak{t}_i,k_i)}(\hat{\tau}_i)|_{\mathfrak{s}}$, we define for every $y \in \mathbf{R}^{d+1}$:

$$(\bar{\varphi}_{x,\bar{x},y} - \mathrm{T}_{|\cdot|_{\mathfrak{s}},x,y}(\bar{\varphi}_{x,\bar{x}}))(\hat{\tau}) := \big(\bar{\varphi}_{x,\bar{x},y}(X^n) - \mathrm{T}_{|n|_{\mathfrak{s}},x,y}(\bar{\varphi}_{x,\bar{x},\cdot}(X^n))\big)$$
$$\prod_i (\bar{\varphi}_{x,\bar{x},y}(\mathcal{I}_{(\mathfrak{t}_i,k_i)}(\hat{\tau}_i)) - \mathrm{T}_{\alpha_i,x,y}(\bar{\varphi}_{x,\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t}_i,k_i)}(\hat{\tau}_i)))$$

Proposition 3.16 *One has for every $\hat{\tau} \in \mathcal{T}^{\mathcal{D}}$:*

$$\varphi^{+}_{\bar{x},x,y}(\hat{\tau}) = \big(\bar{\varphi}_{x,\bar{x},y} - \mathrm{T}_{|\cdot|_{\mathfrak{s}},x,y}(\bar{\varphi}_{x,\bar{x}})\big)(\hat{\tau}).$$

Therefore $\varphi^{+}_{\bar{x},x,\cdot}(\hat{\tau})$ belongs to $\mathcal{H}^{+}_{x}$.

Proof. By multiplicativity, we just need to check this property for X_i and trees of the form $\mathcal{I}_{(\mathfrak{t},k)}(\hat{\tau})$. First one can check that

$$\begin{aligned}\bar{\varphi}_{x,\bar{x},y}(X_i) - \mathrm{T}_{1,x,y}(\bar{\varphi}_{x,\bar{x},\cdot}(X_i)) &= \varphi_{\bar{x},y}(X_i) - \varphi_{\bar{x},x}(X_i)\\ &= (y_i - \bar{x}_i) - (x_i - \bar{x}_i) = y_i - x_i.\end{aligned}$$

From X_i being primitive follows that:

$$\varphi^{+}_{x,\bar{x},y}(X_i) = \big(\varphi_{\bar{x},y} \otimes \varphi^{-}_{x,\bar{x},\bar{x}}\big)\hat{\Delta}^{+} X_i = y_i - x_i.$$

For $\hat{\tau} = \mathcal{I}_{(\mathfrak{t},k)}(\hat{\tau}_1)$, we get

$$\begin{aligned}&(\varphi_{\bar{x},y} \otimes \varphi^{-}_{x,\bar{x},\bar{x}})\hat{\Delta}^{+}\hat{\tau} = (\varphi_{\bar{x},y} \otimes \varphi^{-}_{x,\bar{x},\bar{x}})\hat{\Delta}^{+}\mathcal{I}_{(\mathfrak{t},k)}(\hat{\tau}_1)\\ &= \big(\varphi_{\bar{x},y}\mathcal{I}_{(\mathfrak{t},k)} \otimes \varphi^{-}_{x,\bar{x},\bar{x}}\big)\hat{\Delta}^{+}\hat{\tau}_1 + \sum_{|\ell|_{\mathfrak{s}}<|\hat{\tau}|_{\mathfrak{s}}} \left(\varphi_{\bar{x},y}\frac{X^{\ell}}{\ell!} \otimes \varphi^{-}_{x,\bar{x},\bar{x}}\mathcal{I}_{(\mathfrak{t},k+\ell)}(\hat{\tau}_1)\right)\\ &= \bar{\varphi}_{x,\bar{x},y}(\hat{\tau}) - \sum_{|\ell|_{\mathfrak{s}}<|\hat{\tau}|_{\mathfrak{s}}} \left(\frac{(y-\bar{x})^{\ell}}{\ell!}\mathrm{T}_{|\hat{\tau}|_{\mathfrak{s}}-|\ell|_{\mathfrak{s}},x,\bar{x}}[\bar{\varphi}_{x,\bar{x},\cdot}\mathcal{I}_{(\mathfrak{t},k+\ell)}(\hat{\tau}_1)]\right)\\ &= \bar{\varphi}_{x,\bar{x},y}(\hat{\tau}) - \mathrm{T}_{|\bar{\tau}|_{\mathfrak{s}},x,y}(\bar{\varphi}_{x,\bar{x},\cdot}(\hat{\tau})).\end{aligned}$$

where we have used the identities (3.10) and (3.13). □

In order to get some $\bar{x}$-invariant properties, we need to be more precise in the choice of the characters which is the aim of the next assumption.

Assumption 2 *We assume that the family of characters $(\varphi_{\bar{x}})_{\bar{x}\in\mathbf{R}^{d+1}}$ satisfies:*

$$\begin{cases}\varphi_{\bar{x},y}(X_i) = y_i - \bar{x}_i,\\ \varphi_{\bar{x},y}(\mathcal{I}_{(\mathfrak{t},k)}(\hat{\tau})) = \int_{\mathbf{R}^{d+1}} D^k K_{\mathfrak{t}}(y-z)\varphi_{\bar{x},z}(\hat{\tau})dz\end{cases} \tag{3.14}$$

where $(K_{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{L}}$ is a family of smooth kernels.

Theorem 3.17 *Under the Assumption 2, one gets*

1. *The renormalised character map $\varphi^+_{x,\bar{x}}$ is $\bar{x}$-invariant: $\varphi^+_{x,0} := \varphi^+_{x,\bar{x}}$.*
2. *Bogoliubov's preparation map $\bar{\varphi}_{x,\bar{x}}$ is $\bar{x}$-invariant on planted trees and:*

$$\bar{\varphi}_{x,\bar{x},y}(\mathcal{I}_{(\mathfrak{t},k)}(\widehat{\tau})) = (D^k K_{\mathfrak{t}} * \varphi^+_x(\widehat{\tau}))(y).$$

3. *The counter term map $\varphi^-_{x,\bar{x}}$ is $\bar{x}$-invariant on trees which have a zero node decoration at the root.*

Proof. We first prove the $\bar{x}$-invariance for the renormalised character map $\varphi^+_{x,\bar{x}}$. By the morphism property, we need to check it on X_i and $\mathcal{I}_{(\mathfrak{t},k)}(\widehat{\tau})$. One has:

$$\varphi^+_{x,\bar{x},y}(X_i) = y_i - x_i.$$

Then for $\mathcal{I}_{(\mathfrak{t},k)}(\widehat{\tau})$, one gets

$$\begin{aligned}\bar{\varphi}_{x,\bar{x},\cdot}(\mathcal{I}_{(\mathfrak{t},k)}(\widehat{\tau})) &= \big(\varphi_{\bar{x},\cdot}\mathcal{I}_{(\mathfrak{t},p)} \otimes \varphi^-_{x,\bar{x},\bar{x}}\big)\hat{\Delta}^+\widehat{\tau} \\ &= \Big(D^k K_{\mathfrak{t}} * \varphi_{\bar{x},\cdot} \otimes \varphi^-_{x,\bar{x},\bar{x}}\Big)\hat{\Delta}^+\widehat{\tau} \\ &= \Big(D^k K_{\mathfrak{t}} * \varphi_{\bar{x},\cdot} \otimes \varphi^-_{x,\bar{x},\bar{x}}\Big)\hat{\Delta}^+\widehat{\tau} \\ &= (D^k K_{\mathfrak{t}} * \varphi^+_{x,\bar{x}}(\widehat{\tau}))(\cdot).\end{aligned}$$

We conclude by applying an induction hypothesis on $\varphi^+_{x,\bar{x},\cdot}(\widehat{\tau})$ and by using proposition 3.16. We also proved the formula for $\bar{\varphi}_{x,\bar{x},y}$ and the $\bar{x}$-invariance on planted trees. For $\varphi^-_{x,\bar{x}}$, we use the character property and we consider only planted trees. By the definition of $\varphi^-_{x,\bar{x}}$ which involves $\bar{\varphi}_{x,\bar{x}}$, we conclude from the previous $\bar{x}$-invariance. □

4 Applications

4.1 Link with singular SPDEs

The formalism of decorated trees has been developed originally for singular Stochastic Partial Differential Equations (SPDEs). In this context, a decorated tree is a combinatorial representation of an iterated integral obtained from a perturbative expansion performed using the mild formulation of the equation. Indeed, considering an SPDE of the form:

$$\partial_t u - \Delta u = F(u, \nabla u, \xi), \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d, \tag{4.1}$$

where ξ is a space-time noise, the mild formulation of (4.1) is given by:

$$u = K * (F(u, \nabla u, \xi)),$$

where K is the heat kernel and $*$ denotes space-time convolution. Then, if F is a polynomial, one iterates this equation on u as well as ∇u. The noise ξ is replaced by a mollified noise ξ_ε. If F is not a polynomial, the iteration is performed after its Taylor expansion, which makes appear polynomials in the iterated integrals.

More generally, we suppose given a finite set $\mathfrak{L} = \mathfrak{L}_+ \sqcup \mathfrak{L}_-$, a collection of kernels $(K_{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{L}_+}$ which corresponds to the kernels appearing in the mild formulation and a collection of smooth noises $(\xi_{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{L}_-}$. The scaling $\mathfrak{s} \in \mathbf{N}^{d+1}$ is fixed to be parabolic, $\mathfrak{s} = (2, 1, \ldots, 1)$. Furthermore, it is supposed that (4.1) is locally subcritical. Then, one can construct a subspace $\bar{\mathcal{T}}^{\mathfrak{D}} \subset \mathcal{T}^{\mathfrak{D}}$, which is stable under the coproduct on decorated trees and generated by a subcritical and normal complete rule R. We shall refrain from further details and refer the reader to [BHZ19, Section 5], where those rules have been detailed. The decorated trees in $\bar{\mathcal{T}}^{\mathfrak{D}}$ associated to the equation allow to describe the iterated integrals coming from its perturbative expansion.

The essential point is that $\bar{\mathcal{T}}^{\mathfrak{D}}$ is a subset of $\mathcal{T}^{\mathfrak{D}}$ but not necessarily a subalgebra. Indeed, the rules reflect the distributional products of the equation, which imply that trees cannot be multiplied using the tree product: yielding products that we will not be able to renormalise.

However, the (co)algebraic results given in Section 3 remain valid when one replaces $\mathcal{T}^{\mathfrak{D}}$ by $\bar{\mathcal{T}}^{\mathfrak{D}}$. Note that decorated trees are still abbreviated by $\hat{\tau} \in \bar{\mathcal{T}}^{\mathfrak{D}}$.

Then, the iterated integrals are given by a family of characters, $\mathbf{\Pi}^{(\bar{x})} : \bar{\mathcal{T}}^{\mathfrak{D}} \to \mathscr{H}$, indexed by $\bar{x} \in \mathbf{R}^{d+1}$ and recursively defined by

$$\begin{cases} (\mathbf{\Pi}^{(\bar{x})}\mathbf{1})(y) = 1, \qquad (\mathbf{\Pi}^{(\bar{x})}X_i)(y) = y_i - \bar{x}_i, \\ (\mathbf{\Pi}^{(\bar{x})}\mathcal{I}_{(\mathfrak{t},k)}(\hat{\tau}))(y) = \displaystyle\int_{\mathbf{R}^{d+1}} D^k K_{\mathfrak{t}}(y-z)(\mathbf{\Pi}^{(\bar{x})}\hat{\tau})(z)dz, \quad \mathfrak{t} \in \mathfrak{L}_+ \\ (\mathbf{\Pi}^{(\bar{x})}\mathcal{I}_{(\mathfrak{t},k)}(\mathbf{1}))(y) = D^k\xi_{\mathfrak{t}}(y), \quad \mathfrak{t} \in \mathfrak{L}_-. \end{cases} \tag{4.2}$$

We add as an extra assumption that edges of type $\mathfrak{t} \in \mathfrak{L}_-$ can only appear as terminal edges. The character

$$\mathbf{\Pi} := \mathbf{\Pi}^{(0)} \tag{4.3}$$

has been introduced in [Hai14] where it was identified as the main character for the construction of the model in [BHZ19]. However, we would like to consider instead $\mathbf{\Pi}^{(\bar{x})}$, where a priori recentering of the polynomials around $\bar{x} \in \mathbf{R}^{d+1}$ is allowed, and see how the main objects used in the theory of SPDEs behave toward this change.

As presented in the previous section, we are given a scaling $\mathfrak{s}$ together with a map $|\cdot|_{\mathfrak{s}} : \mathfrak{L} \to \mathbf{R}$. The latter depends on the analytical assumptions put on the kernels and noises. It is expect that $|\cdot|_{\mathfrak{s}} : \mathfrak{L}_+ \to \mathbf{R}_+$ corresponds to Schauder estimates for improvement of regularity and that $|\cdot|_{\mathfrak{s}} : \mathfrak{L}_+ \to \mathbf{R}_-$ encodes the singularity of the noises when the mollification is removed. We consider the space $\bar{\mathcal{T}}_+^{\mathfrak{D}}$ as not being a subspace of $\bar{\mathcal{T}}^{\mathfrak{D}}$, but as a subspace of

$$\bar{\mathcal{T}}_+^{\mathfrak{D}} \subset \hat{\mathcal{T}}_+^{\mathfrak{D}} = \Big\{ X^k \prod_{i=1}^n \hat{\mathcal{J}}_{(\mathfrak{t}_i,p_i)}(\hat{\tau}_i), \hat{\tau}_i \in \bar{\mathcal{T}}^{\mathfrak{D}} \Big\}. \tag{4.4}$$

Here the symbol $\hat{\mathcal{J}}_{(\mathfrak{t},p)}$ instead of $\mathcal{I}_{(\mathfrak{t},p)}$ is used for the edges outgoing of the root. We denote by $\mathfrak{i}_+$ the injection from $\hat{\mathcal{T}}_+^{\mathfrak{D}}$ to $\mathcal{T}^{\mathfrak{D}}$. Elements of $\mathcal{T}_+^{\mathfrak{D}}$ are defined by applying the projection map π_+ which is now considered from $\hat{\mathcal{T}}_+^{\mathfrak{D}}$ to $\mathcal{T}_+^{\mathfrak{D}}$. The symbol $\mathcal{J}_{(\mathfrak{t},p)}$ will be a short hand notation for $\pi_+ \circ \hat{\mathcal{J}}_{(\mathfrak{t},p)}$. The map $\mathbf{\Pi}^{(\bar{x})}$ can also be defined on $\hat{\mathcal{T}}_+^{\mathfrak{D}}$ by setting:

$$\mathbf{\Pi}^{(\bar{x})}\hat{\mathcal{J}}_{(\mathfrak{t},p)}(\hat{\tau}) := \mathbf{\Pi}^{(\bar{x})}\mathcal{I}_{(\mathfrak{t},p)}(\hat{\tau}).$$

The main reason for marking this difference is that $\bar{\mathcal{T}}^{\mathfrak{D}}$, in general, is not an algebra. Indeed, the construction of $\bar{\mathcal{T}}^{\mathfrak{D}}$ is constrained by the distributional product appearing on the right-hand side of (4.1). On the other hand, $\bar{\mathcal{T}}_+^{\mathfrak{D}}$ is always an algebra.

One of the main achievements of Hairer's theory of regularity structures is to provide a Taylor expansion of the solution u of (4.1):

$$u(y) = u(x) + \sum_{\hat{\tau}\in\bar{\mathcal{T}}^{\mathfrak{D}}_{\text{eq}}} \Upsilon[\hat{\tau}](x)(\Pi_x\hat{\tau})(y) + R(x,y),$$

where $\bar{\mathcal{T}}^{\mathfrak{D}}_{\text{eq}} \subset \bar{\mathcal{T}}^{\mathfrak{D}}$ contains trees generated by the perturbative expansion and $\Upsilon[\hat{\tau}](x)$ are coefficients which may depend on u and its derivatives ∇u. The map Π_x is deduced from $\mathbf{\Pi}^{(0)} = \mathbf{\Pi}$ in Definition 4.1 below. Then, one can define a Regularity Structure $(\bar{\mathcal{T}}^{\mathfrak{D}}, G)$ where

- $\bar{\mathcal{T}}^{\mathfrak{D}} = \bigoplus_{\alpha\in A}\bar{\mathcal{T}}^{\mathfrak{D}}_\alpha$ is a graded space with $A \subset \mathbf{R}$ bounded from below and locally finite. For any element $\hat{\tau} \in \bar{\mathcal{T}}^{\mathfrak{D}}_\alpha$, one has $|\hat{\tau}|_{\mathfrak{s}} = \alpha$. When $\hat{\tau} \in \bar{\mathcal{T}}^{\mathfrak{D}}$, $\|\hat{\tau}\|_\alpha$ denotes the norm of its component in $\bar{\mathcal{T}}^{\mathfrak{D}}_\alpha$.
- G is a structure group of continuous linear operators acting on $\bar{\mathcal{T}}^{\mathfrak{D}}$ such that, for every $\Gamma \in G$, every $\alpha \in A$ and $\hat{\tau} \in \bar{\mathcal{T}}^{\mathfrak{D}}_\alpha$, one has

$$\Gamma\hat{\tau} - \hat{\tau} \in \bigoplus_{\beta<\alpha}\bar{\mathcal{T}}^{\mathfrak{D}}_\beta.$$

Elements in G are maps of the form Γ_g where $g : \bar{\mathcal{T}}_+^{\mathfrak{D}} \to \mathbf{R}$ is a character and

$$\Gamma_g = (\mathrm{id} \otimes g)\hat{\Delta}^+.$$

One of the main definitions in [Hai14, BHZ19] makes precise the structures behind the construction of the map Π_x.

Definition 4.1 Given the linear map $\mathbf{\Pi}\colon \bar{\mathcal{T}}^{\mathfrak{D}} \to \mathcal{H}$, we define for all $z, \bar{z} \in \mathbf{R}^{d+1}$

- a linear map $\Pi_z\colon \bar{\mathcal{T}}^{\mathfrak{D}} \to \mathcal{H}$ and a character $f_z : \bar{\mathcal{T}}_+^{\mathfrak{D}} \to \mathcal{H}$ by

$$\Pi_z = (\mathbf{\Pi} \otimes f_z)\hat{\Delta}^+ \;, \qquad f_z = \big(\mathbf{\Pi}\tilde{\mathcal{A}}_+\cdot\big)(z) \;, \tag{4.5}$$

where $\tilde{\mathcal{A}}_+$ is the positive twisted antipode given in (4.9) below.

- a linear map $\Gamma_{z\bar{z}}\colon \bar{\mathcal{T}}^{\mathfrak{D}} \to \bar{\mathcal{T}}^{\mathfrak{D}}$ and a character $\gamma_{z\bar{z}} : \bar{\mathcal{T}}_+^{\mathfrak{D}} \to \mathcal{H}$

$$\Gamma_{z\bar{z}} = (\mathrm{id} \otimes \gamma_{z\bar{z}})\hat{\Delta}^+, \quad \gamma_{z\bar{z}} = (f_z\mathcal{A}_+ \otimes f_{\bar{z}})\bar{\Delta}^+ \ . \tag{4.6}$$

Then under certain favourable conditions on the map $\mathbf{\Pi}$ which correspond to the definition given in (4.2), (Π, Γ) is an admissible model satisfying the following:

- Algebraic properties:

$$\Gamma_{xx} = \mathrm{id}, \quad \Gamma_{xy} \circ \Gamma_{yz} = \Gamma_{xz}, \quad \Pi_y = \Pi_x \circ \Gamma_{xy} \tag{4.7}$$

- Analytical bounds: Let $\hat{\tau} \in \bar{\mathcal{T}}^{\mathfrak{D}}$ and $\ell = |\hat{\tau}|_{\mathfrak{s}}$. For every compact set $\mathfrak{K} \subset \mathbf{R}^{d+1}$, we assume the existence of a constant $C_{\ell,\mathfrak{K}}$ such that the bounds

$$|(\Pi_x\hat{\tau})(y)| \leq C_{\ell,\mathfrak{K}}\|\hat{\tau}\|_\ell \,\|x-y\|_{\mathfrak{s}}^\ell, \qquad \|\Gamma_{xy}\hat{\tau}\|_m \leq C_{\ell,\mathfrak{K}}\|\hat{\tau}\|_\ell \,\|x-y\|_{\mathfrak{s}}^{\ell-m} \ , \tag{4.8}$$

  hold uniformly over all $x, y \in \mathfrak{K}$, all $m \in A$ with $m < \ell$ and all $\hat{\tau}$ such that $|\hat{\tau}|_{\mathfrak{s}} = \ell$

From [BHZ19, Prop. 6.3] it follows that there exists a unique algebra morphism $\tilde{\mathcal{A}}_+\colon \bar{\mathcal{T}}_+^{\mathfrak{D}} \to \hat{\mathcal{T}}_+^{\mathfrak{D}}$, called the "positive twisted antipode", such that $\tilde{\mathcal{A}}_+X_i = -X_i$ and furthermore for all $\mathcal{J}_{(\mathfrak{t},k)}(\hat{\tau}) \in \bar{\mathcal{T}}_+^{\mathfrak{D}}$

$$\tilde{\mathcal{A}}_+\mathcal{J}_{(\mathfrak{t},k)}(\hat{\tau}) = -\sum_{|\ell|_{\mathfrak{s}}<|\mathcal{J}_{(\mathfrak{t},k)}(\hat{\tau})|_{\mathfrak{s}}} \frac{(-X)^\ell}{\ell!}\hat{\mathcal{M}}_+(\hat{\mathcal{J}}_{(\mathfrak{t},k+\ell)} \otimes \tilde{\mathcal{A}}_+)\hat{\Delta}^+\hat{\tau} \ , \tag{4.9}$$

where $\hat{\mathcal{M}}_+ : \hat{\mathcal{T}}_+^{\mathfrak{D}} \otimes \hat{\mathcal{T}}_+^{\mathfrak{D}} \to \hat{\mathcal{T}}_+^{\mathfrak{D}}$ is the tree product on $\hat{\mathcal{T}}_+^{\mathfrak{D}}$. Then we consider the character φ defined for every $\bar{x} \in \mathbf{R}^{d+1}$ by:

$$\varphi_{\bar{x}}(\hat{\tau}) = (\mathbf{\Pi}^{(\bar{x})}\hat{\tau}). \tag{4.10}$$

The main characters used for defining the model associated to a SPDE are:

$$f_x^{(\bar{x})} = (\Pi^{(\bar{x})}\tilde{\mathcal{A}}_+\cdot)(x), \quad \Pi_x^{(\bar{x})} = \left(\mathbf{\Pi}^{(\bar{x})} \otimes f_x^{(\bar{x})}\right)\hat{\Delta}^+$$

Proposition 4.2 *Under the assumption* (4.10)*, one gets:*

$$\Pi_x^{(\bar{x})} = \varphi_{x,\bar{x}}^+, \quad f_x^{(\bar{x})} = \varphi_{x,\bar{x},\bar{x}}^-.$$

Proof. The fact that $f_x^{(\bar{x})} = \varphi_{x,\bar{x},\bar{x}}^-$ comes from Proposition 3.15. Then, we conclude using Definition 3.11. □

Corollary 4.3 *The model* (Π, Γ) *is invariant under translations of the monomials. In the sense that for every* $\bar{x} \in \mathbf{R}^{d+1}$*:*

$$\Pi_x = \Pi_x^{(\bar{x})}, \quad \Gamma_{xy} = \Gamma_{xy}^{(\bar{x})}.$$

Proof. For $\Pi_x^{(\bar{x})}$, the invariance follows from Theorems 3.17 and 4.2. Then for Γ_{xy}, we proceed by induction through the formula introduced in [Bru18, Prop. 3.13]:

$$\Gamma_{xy}\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau}) = \mathscr{I}_{(\mathfrak{t},k)}(\Gamma_{xy}\widehat{\tau}) + \sum_{|\ell|_{\mathfrak{s}} \leq |\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})|_{\mathfrak{s}}} \frac{(X + (y-x)\mathbf{1})^{\ell}}{\ell!} (\Pi_x \mathscr{I}_{(\mathfrak{t},k)}(\Gamma_{xy}\widehat{\tau}))(y).$$

□

We are also able to recover the recursive formulation proposed as a definition in [Hai14] for the maps Π_x and $f_x^{(\bar{x})}$.

Proposition 4.4 *The map Π_x is given for $\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})$ with $|\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})|_{\mathfrak{s}} = \alpha$ by*

$$\begin{aligned}(\Pi_x \mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau}))(y) &= (D^k K_{\mathfrak{t}} * \Pi_x \widehat{\tau})(y) \\ &\quad - \sum_{|\ell|_{\mathfrak{s}} \leq \alpha} \frac{(y-x)^{\ell}}{\ell!} (D^{k+\ell} K_{\mathfrak{t}} * \Pi_x \widehat{\tau})(x)\end{aligned} \tag{4.11}$$

and the map $f_x^{(\bar{x})}$ is given for $\mathscr{J}_{(\mathfrak{t},k)}(\widehat{\tau})$ with $\alpha = |\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})|_{\mathfrak{s}} > 0$ by

$$f_x^{(\bar{x})}(\mathscr{J}_{(\mathfrak{t},k)}(\widehat{\tau})) = -\sum_{|\ell|_{\mathfrak{s}} \leq \alpha} \frac{(\bar{x}-x)^{\ell}}{\ell!} (D^{k+\ell} K_{\mathfrak{t}} * \Pi_x \widehat{\tau})(x). \tag{4.12}$$

Proof. Indeed, from Theorem 3.17, one has an explicit formula for the preparation map $\bar{\varphi}_{x,\bar{x}}$ given by

$$\bar{\varphi}_{x,\bar{x}}(\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})) = D^k K_{\mathfrak{t}} * \varphi_x^+(\widehat{\tau}) = D^k K_{\mathfrak{t}} * \Pi_x(\widehat{\tau}).$$

Therefore, for $\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})$ with $|\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})|_{\mathfrak{s}} = \alpha$

$$\begin{aligned}(\Pi_x \mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau}))(y) &= (\mathrm{id} - \mathrm{T}_{\alpha,x,\cdot})(\bar{\varphi}_{x,\bar{x}}(\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})))(y) \\ &= (\mathrm{id} - \mathrm{T}_{\alpha,x,\cdot})\Big(D^k K_{\mathfrak{t}} * \Pi_x(\widehat{\tau})\Big)(y) \\ &= (D^k K_{\mathfrak{t}} * \Pi_x \widehat{\tau})(y) - \sum_{|\ell|_{\mathfrak{s}} \leq \alpha} \frac{(y-x)^{\ell}}{\ell!} (D^{k+\ell} K_{\mathfrak{t}} * \Pi_x \widehat{\tau})(x).\end{aligned}$$

Then, for $\mathscr{J}_{(\mathfrak{t},k)}(\widehat{\tau})$ with $\alpha = |\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})|_{\mathfrak{s}} > 0$, we get:

$$\begin{aligned}f_x^{(\bar{x})}(\mathscr{J}_{(\mathfrak{t},k)}(\widehat{\tau})) &= -\mathrm{T}_{\alpha,x,\bar{x}}(\bar{\varphi}_{x,\bar{x}}(\mathfrak{i}_+(\mathscr{J}_{(\mathfrak{t},k)}(\widehat{\tau})))) \\ &= -\mathrm{T}_{\alpha,x,\bar{x}}(\bar{\varphi}_{x,\bar{x}}(\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau}))) \\ &= \sum_{|\ell|_{\mathfrak{s}} \leq \alpha} \frac{(\bar{x}-x)^{\ell}}{\ell!} (D^{k+\ell} K_{\mathfrak{t}} * \Pi_x \widehat{\tau})(x).\end{aligned}$$

One recovers for $\bar{x} = 0$ the formula given in [BHZ19, Lemma 6.10]. □

Remark 4.5 If we take $\bar{x} = x$, we obtain a simple formula for $f_x^{(\bar{x})}$:

$$f_x^{(x)}(\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})) = -\mathbf{1}_{|\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})|_{\mathfrak{s}}>0} \Big(D^k K_{\mathfrak{t}} * \Pi_x \widehat{\tau} \Big)(x).$$

When $\bar{x} = 0$, which corresponds to the case considered in [Hai14, BHZ19], one can fix $x = 0$ and obtains

$$f_0(\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})) = -\mathbf{1}_{|\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau})|_{\mathfrak{s}}>0} \Big(D^k K_{\mathfrak{t}} * \Pi_0 \widehat{\tau} \Big)(0).$$

If we consider only this case, some simplifications can be introduced in the algebraic structures. The Bogoliubov recursion given in (3.11) is the same except that now one can replace the map Δ^+ by changing its value on polynomials:

$$\Delta^+ X_i = X_i \otimes \mathbf{1},$$

This will also change the definition of the space $\mathcal{T}_+^{\mathcal{D}}$ which will no longer contain the polynomials. One gets the following definition:

$$\mathcal{T}_+^{\mathcal{D}} = \{ \prod_{i=1}^n \mathscr{I}_{(\mathfrak{t}_i,p_i)}(\widehat{\tau}_i), |\mathscr{I}_{(\mathfrak{t}_i,p_i)}(\widehat{\tau}_i)|_{\mathfrak{s}} > 0, \widehat{\tau}_i \in \bar{\mathcal{T}}^{\mathcal{D}} \}.$$

Then the coaction $\hat{\Delta}^+ : \bar{\mathcal{T}}^{\mathcal{D}} \to \bar{\mathcal{T}}^{\mathcal{D}} \otimes \bar{\mathcal{T}}_+^{\mathcal{D}}$ is given by:

$$\begin{aligned} \hat{\Delta}^+ X_i &= X_i \otimes \mathbf{1}, \\ \hat{\Delta}^+ \mathscr{I}_{(\mathfrak{t},p)}(\widehat{\tau}) &= (\mathscr{I}_{(\mathfrak{t},p)} \otimes \mathrm{id}) \hat{\Delta}^+ \widehat{\tau} + \sum_{\ell \in \mathbf{N}^{d+1}} \frac{X^\ell}{\ell!} \otimes \mathscr{I}_{(\mathfrak{t},p+\ell)}(\widehat{\tau}) \end{aligned} \tag{4.13}$$

We keep the structure of the deformation in this definition, but now X_i is not primitive. The main change occurs for the coproduct $\bar{\Delta}^+$ where all the deformation at the edge adjacent to the root is removed.

$$\bar{\Delta}^+ \mathscr{I}_{(\mathfrak{t},p)}(\widehat{\tau}) = (\mathscr{I}_{(\mathfrak{t},p)} \otimes \mathrm{id}) \hat{\Delta}^+ \widehat{\tau} + \mathbf{1} \otimes \mathscr{I}_{(\mathfrak{t},p)}(\widehat{\tau}). \tag{4.14}$$

Moreover, the Hopf algebra $\mathcal{T}_+^{\mathcal{D}}$ is now connected and one gets

$$\mathcal{A}_+ X_i = -X_i, \quad \mathcal{A}_+ \mathscr{I}_{(\mathfrak{t},p)}(\widehat{\tau}) = -\mathcal{M}(\mathscr{I}_{(\mathfrak{t},p)} \otimes \mathcal{A}_+) \bar{\Delta}^+ \widehat{\tau}. \tag{4.15}$$

Even the twisted antipode $\tilde{\mathcal{A}}_+$ is simplified in this context:

$$\tilde{\mathcal{A}}_+ \mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau}) = -\hat{\mathcal{M}}_+ (\hat{\mathscr{I}}_{(\mathfrak{t},k)} \otimes \tilde{\mathcal{A}}_+) \hat{\Delta}^+ \widehat{\tau}. \tag{4.16}$$

In certain cases, it will also coincide with the antipode. Indeed, one can have:

$$\hat{\Delta}^+ \widehat{\tau} = \sum_{(\widehat{\tau})} \widehat{\tau}' \otimes \widehat{\tau}'', \quad |\mathscr{I}_{(\mathfrak{t},k)}(\widehat{\tau}')|_{\mathfrak{s}} \geq 0.$$

In fact, this case is almost the general case. Indeed, it happens for branched rough paths, generalised KPZ equations and on every equation when the space $\bar{\mathscr{T}}^{\mathfrak{D}}_{\text{eq}}$ is a positive sector, that is, a subspace invariant by G and where the minimum degree is 0. For a negative sector, one can always remove the negative part with a generalised Da Prato–Debussche trick see [BCCH17, Section 5] and work on a positive sector.

We can reformulate (3.11) as

$$\begin{aligned} \bar{\varphi}(\widehat{\tau})(y) &= \varphi(\widehat{\tau})(y) + \sum\nolimits_{(\widehat{\tau})}^{+} \varphi(\widehat{\tau}')(y)\varphi^{-}(\widehat{\tau}'')(y) \\ \varphi^{-}(\widehat{\tau})(y) &= -\mathrm{T}_{|\widehat{\tau}|_{\mathfrak{s}},0,y}(\bar{\varphi}(\widehat{\tau})) \\ \varphi^{+}(\cdot)(y) &= \varphi(\cdot)(y) \star \varphi^{-}(\cdot)(0) = \big(\varphi(\cdot)(y) \otimes \varphi^{-}(\cdot)(0)\big)\hat{\Delta}^{+} \end{aligned} \tag{4.17}$$

where

$$\varphi = \mathbf{\Pi}, \quad \varphi^{+} = \Pi_0, \quad \varphi^{-}(\cdot)(x) = f_x.$$

Note the shift in notation due to the simplifications implied by the choice of parameters. Then, for $x = 0$, the character $\varphi^{-} := \varphi^{-}(\cdot)(0)$ is given by

$$\varphi^{-}(\widehat{\tau})(0) = -\mathrm{T}_{|\widehat{\tau}|_{\mathfrak{s}},0,0}(\bar{\varphi}(\widehat{\tau})) = \mathrm{ev}_0(\bar{\varphi}(\widehat{\tau})),$$

where ev_x is the evaluation at the point x. These simplifications have been observed in [BS20, Remark 2.11] and make sense when one can just look at Π_0 for the convergence of the model. This is true for a random model whose law is invariant by translation, in the sense that $\Pi_x(\widehat{\tau})$ and $\Pi_0(\widehat{\tau})$ have the same law.

We can go even further in the simplifications if $y = 0$. Then, one can work directly with the Connes–Kreimer coproduct as the deformation will disappear when we apply $\varphi(\cdot)(0)$.

Remark 4.6 The construction relies on $\mathbf{\Pi}$ being a character. When in Regularity Structures, we renormalise characters with a suitable map $M : \bar{\mathscr{T}}^{\mathfrak{D}} \to \bar{\mathscr{T}}^{\mathfrak{D}}$, we construct $\mathbf{\Pi}^{M}$ as a character on $\bar{\mathscr{T}}^{\mathfrak{D}}_{+}$ but on $\bar{\mathscr{T}}^{\mathfrak{D}}$ it will not: we need to renormalise ill-defined distributional product and so the multiplicativity will be lost. Therefore, the construction is still valid for $\mathbf{\Pi}^{M}$ viewed as a character on $\bar{\mathscr{T}}^{\mathfrak{D}}_{+}$. It has a meaning for φ_{-} but not for φ_{+}. We will expand this construction in the next section.

4.2 Negative Renormalisation

Let $\hat{\mathscr{T}}^{\mathfrak{D}}_{-}$ the free commutative algebra generated by $\bar{\mathscr{T}}^{\mathfrak{D}}$. We denote by $\cdot$ the forest product associated to this algebra. The empty forest is given by $\mathbf{1}_1$. Elements of $\hat{\mathscr{T}}^{\mathfrak{D}}_{-}$ are of the form $(F, \mathfrak{n}, \mathfrak{e})$ where F is a forest. The forest product is defined by:

$$(F, \mathfrak{n}, \mathfrak{e}) \cdot (G, \bar{\mathfrak{n}}, \bar{\mathfrak{e}}) = (F \cdot G, \bar{\mathfrak{n}} + \mathfrak{n}, \bar{\mathfrak{e}} + \mathfrak{e}),$$

where the sums $\bar{\mathfrak{n}} + \mathfrak{n}$ and $\bar{\mathfrak{e}} + \mathfrak{e}$ mean that decorations defined on one of the forests are extended to the disjoint union by setting them to vanish on the other forest.

Then we set $\bar{\mathcal{T}}_-^{\mathfrak{D}} = \hat{\mathcal{T}}_-^{\mathfrak{D}}/\mathcal{B}_+^{\mathfrak{D}}$ where $\mathcal{B}_+^{\mathfrak{D}}$ is the ideal of $\hat{\mathcal{T}}_-^{\mathfrak{D}}$ generated by $\{\hat{\tau} \in B_\circ \;:\; |\hat{\tau}|_{\mathfrak{s}} \geq 0\}$, where $B_\circ \subset \mathcal{R}\mathcal{T}^{\mathfrak{D}}$ and $\langle B_\circ \rangle = \bar{\mathcal{T}}^{\mathfrak{D}}$. Then, one defines in [BHZ19], a coaction $\hat{\Delta}^- : \bar{\mathcal{T}}^{\mathfrak{D}} \to \bar{\mathcal{T}}_-^{\mathfrak{D}} \otimes \bar{\mathcal{T}}^{\mathfrak{D}}$ which is a deformation of an extraction-contraction coproduct in the same spirit as $\hat{\Delta}^+$. By multiplicativity, one extends it to a coaction $\hat{\Delta}^- : \hat{\mathcal{T}}_-^{\mathfrak{D}} \to \bar{\mathcal{T}}_-^{\mathfrak{D}} \otimes \hat{\mathcal{T}}_-^{\mathfrak{D}}$.

Then one can turn this map into a coproduct $\bar{\Delta}^- : \bar{\mathcal{T}}_-^{\mathfrak{D}} \to \bar{\mathcal{T}}_-^{\mathfrak{D}} \otimes \bar{\mathcal{T}}_-^{\mathfrak{D}}$ and obtains a connected Hopf algebra for $\bar{\mathcal{T}}_-^{\mathfrak{D}}$ endowed with this coproduct and the forest product see [BHZ19, Prop. 5.35]. The main difference here is that we do not consider extended decorations but the results for the Hopf algebra are the same as in [BHZ19]. The twisted antipode is given in [BHZ19, Prop. 6.6]

Proposition 4.7 *There exists a unique algebra morphism $\tilde{\mathcal{A}}_- : \bar{\mathcal{T}}_-^{\mathfrak{D}} \to \hat{\mathcal{T}}_-^{\mathfrak{D}}$, that we call the "negative twisted antipode", such that for $\hat{\tau} \in \bar{\mathcal{T}}_-^{\mathfrak{D}} \cap \ker \mathbf{1}_1^\star$*

$$\tilde{\mathcal{A}}_- \hat{\tau} = -\hat{\mathcal{M}}_- (\tilde{\mathcal{A}}_- \otimes \mathrm{id})(\hat{\Delta}^- \mathrm{i}_- \hat{\tau} - \hat{\tau} \otimes \mathbf{1}_1), \tag{4.18}$$

where i_- is the injection from $\bar{\mathcal{T}}_-^{\mathfrak{D}}$ to $\hat{\mathcal{T}}_-^{\mathfrak{D}}$ and $\hat{\mathcal{M}}_-$ is the forest product between elements of $\hat{\mathcal{T}}_-^{\mathfrak{D}}$.

Remark 4.8 The formalism described here corresponds to the one of Remark 2.9 where $H = \bar{\mathcal{T}}_-^{\mathfrak{D}}$ and $\hat{H} = \hat{\mathcal{T}}_-^{\mathfrak{D}}$. The fact that we get a connected Hopf algebra and that we do not see any polynomials in the reduced coaction comes from the definition of $\mathcal{B}_+^{\mathfrak{D}}$ which contains all the polynomials X^n. Therefore, after quotiening by $\mathcal{B}_+^{\mathfrak{D}}$, they do not belong to $\bar{\mathcal{T}}_-^{\mathfrak{D}}$.

It remains to describe the target space where the Birkhoff factorisation takes place. We denote by $\mathfrak{X}$ the space of stationary processes $X : \Omega \to \mathcal{C}^\infty$ over an underlying probability space $(\Omega, \mathcal{F}, P)$. This means that the laws of $X(z)$ and $X(0)$ are equal when $X \in \mathfrak{X}$ and $z \in \mathbf{R}^{d+1}$. We define $\mathfrak{N} = T(\mathfrak{X})$ as the space of symmetrised tensors over $\mathfrak{X}$. We consider the following splitting:

$$\mathfrak{N} = \mathfrak{N}_- \oplus \mathfrak{N}_+,$$

where $\mathfrak{N}_-$ is the space of constants and $\mathfrak{N}_+$ is the subspace of $\mathfrak{N}$ such that each element F satisfies:

$$\tilde{\mathbb{E}}(F) = 0,$$

where $\tilde{\mathbb{E}} : T(\mathfrak{X}) \to \mathbf{R}$ is defined on $f_1 \otimes \cdots \otimes f_n$ by

$$\tilde{\mathbb{E}}(f_1 \otimes \cdots \otimes f_n) = \prod_{i=1}^n \mathbb{E}(f_i)$$

and $\mathbb{E}$ denotes the expectation over the underlying probability space. Then we consider the characters $\psi : \hat{\mathcal{T}}_-^{\mathfrak{D}} \to \mathfrak{N}$, $\psi_+ : \hat{\mathcal{T}}_-^{\mathfrak{D}} \to \mathfrak{N}$ and $\psi_- : \bar{\mathcal{T}}_-^{\mathfrak{D}} \to \mathfrak{N}_-$ as

$$\psi = \mathbf{\Pi}, \quad \psi_- = \tilde{\mathbb{E}}(\mathbf{\Pi} \tilde{\mathcal{A}}_- \cdot)(0), \quad \psi_+ = \psi_- \star \psi = (\psi_- \otimes \psi)\hat{\Delta}^-,$$

where $\mathbf{\Pi}$ is extended multiplicatively to $\hat{\mathcal{T}}_-^{\mathfrak{D}}$. The choice of the point 0 in the definition of ψ_- is not arbitrary since we suppose that $\mathbf{\Pi}$ takes values in $\mathfrak{X}$. One can show that ψ_+ is taking values in $\mathfrak{N}_+$ see [BHZ19, Theorem 6.18]. With these definitions, one can check that $\tilde{\mathbb{E}}$ is a Rota–Baxter map in the sense of the Remark 2.10 and one gets for $\widehat{\tau} \in \hat{\mathcal{T}}_-^{\mathfrak{D}}$ and $\widehat{\tau}_1 \in \bar{\mathcal{T}}_-^{\mathfrak{D}}$:

$$
\begin{aligned}
\psi_-(\widehat{\tau}_1) = -\tilde{\mathbb{E}}\big(\bar{\psi}(\mathrm{i}_-(\widehat{\tau}_1))\big), \quad \psi_+(\widehat{\tau}) = \Big(\mathrm{id} - \tilde{\mathbb{E}}\Big)\big(\bar{\psi}(\widehat{\tau})\big) \\
\bar{\psi}(\widehat{\tau}) = \psi(\widehat{\tau}) + \sum_{(\widehat{\tau})}^{-} \psi_-(\widehat{\tau}')\psi(\widehat{\tau}'').
\end{aligned}
\tag{4.19}
$$

where the Sweedler's notation is used for the modified reduced coaction:

$$
\Delta^-_{\text{red}}\widehat{\tau} = \sum_{(\widehat{\tau})}^{-} \widehat{\tau}' \otimes \widehat{\tau}''.
$$

The character ψ_- plays a central role in the definition of the renormalised model $\hat{\Pi}_x$ which is given in [BHZ19, Section 6.3] by:

$$
\hat{\Pi}_x = (\mathbf{\Pi} M \otimes f_x M)\hat{\Delta}^+, \quad M = (\psi_- \otimes \mathrm{id})\hat{\Delta}^-.
$$

The fact that this definition gives again a model relies on the cointeraction (see [BHZ19, Theorem 5.37]) between the two Hopf algebras $\hat{\mathcal{T}}_+^{\mathfrak{D}}$ and $\bar{\mathcal{T}}_-^{\mathfrak{D}}$ obtained when one adds extended decorations. This cointeraction has been observed on similar structures without any decorations in [CEM11]. At the level of the coaction, it reads:

$$
\mathcal{M}^{(13)(2)(4)}\Big(\hat{\Delta}^- \otimes \hat{\Delta}^-\Big)\hat{\Delta}^+ = \Big(\mathrm{id} \otimes \hat{\Delta}^+\Big)\hat{\Delta}^-,
\tag{4.20}
$$

where

$$
\mathcal{M}^{(13)(2)(4)}(\widehat{\tau}_1 \otimes \widehat{\tau}_2 \otimes \widehat{\tau}_3 \otimes \widehat{\tau}_4) = \widehat{\tau}_1 \cdot \widehat{\tau}_3 \otimes \widehat{\tau}_2 \otimes \widehat{\tau}_4.
$$

This cointeraction can be seen also at the level of the characters. We will consider two products $\star$ and $*$ given by:

$$
\psi_- \star \psi = (\psi_- \otimes \psi)\hat{\Delta}^-, \quad \varphi * \varphi_x^- = (\varphi \otimes \varphi_x^-)\hat{\Delta}^+,
$$

where $\psi_- : \bar{\mathcal{T}}_-^{\mathfrak{D}} \to \mathbf{R}$, $\psi : \hat{\mathcal{T}}_-^{\mathfrak{D}} \to \mathfrak{N}$, $\varphi : \bar{\mathcal{T}}^{\mathfrak{D}} \to \mathcal{H}$ and $\varphi_x^- : \bar{\mathcal{T}}_+^{\mathfrak{D}} \to \mathcal{H}_x^+$ are characters. Here, we will consider that $\mathcal{H} = \mathfrak{X}$ and ψ is viewed as the multiplicative extension of φ from $\mathfrak{X}$ to $\mathfrak{N}$. Then, we get by using (4.20) on $\bar{\mathcal{T}}^{\mathfrak{D}}$

$$
\begin{aligned}
(\psi_- \star \varphi) * (\psi_- \star \varphi_x^-) &= \psi_- \star (\varphi * \varphi_x^-) \\
&= \psi_- \star \varphi_x^+ = \psi_+ * (\psi_- \star \varphi_x^-),
\end{aligned}
$$

The cointeraction is essential for writing a simple formula for the renormalised model in [BHZ19, Theorem 6.16]:

$$
\hat{\Pi}_x = \Pi_x M.
\tag{4.21}
$$

In fact, one can give a joint Birkhoff decomposition based on the skew product of characters. We denote the group of character associated to $\bar{\mathcal{T}}_-^{\mathfrak{D}}$ (resp. $\bar{\mathcal{T}}_+^{\mathfrak{D}}$) by $\mathcal{G}_-$ (resp. $\mathcal{G}_+$). Then, we denote by $\mathcal{G}_\pm$ the semi-direct product $\mathcal{G}_- \ltimes \mathcal{G}_+$ with group multiplication:

$$(g_1, f_1)(g_2, f_2) = (g_1 \bar{\star} g_2, f_1 \bar{*} (g_1 \bar{\star} f_2)), \quad g_i, \in \mathcal{G}_-,\ f_i \in \mathcal{G}_+,$$

where $\bar{\star}$ is the convolution product associated to $\bar{\Delta}^-$ and $\bar{*}$ is the convolution product associated to $\bar{\Delta}^+$. Given a character $\varphi : \bar{\mathcal{T}}^{\mathfrak{D}} \to \mathfrak{X}$, we consider the following lift:

$$\varphi \to (\psi_-, \psi_+), \quad \psi_+ = \psi_- * \psi,$$

where (ψ_-, ψ_+) is viewed as a character on $\bar{\mathcal{T}}_-^{\mathfrak{D}} \hat{\otimes} \hat{\mathcal{T}}_-^{\mathfrak{D}}$. We can also use another lift of φ:

$$\varphi \to (\psi, \varphi_x^-),$$

where (ψ, φ_x^-) is viewed as a character on $\hat{\mathcal{T}}_-^{\mathfrak{D}} \hat{\otimes} \bar{\mathcal{T}}_+^{\mathfrak{D}}$. Then, one can write:

$$(\psi_+, \hat{\Pi}_x) = (\psi_-, \psi_+)(\psi, \varphi_x^-) = (\psi_- \star \psi, \psi_+ * (\psi_- \star \varphi_x^-)), \quad \hat{\Pi}_x = \psi_- \star \varphi_x^-,$$

where the maps ψ_+ and $\hat{\Pi}_x$ are considered on $\bar{\mathcal{T}}^{\mathfrak{D}}$. Therefore, the term $(\psi_+, \hat{\Pi}_x)$ is not a character on $\hat{\mathcal{T}}_-^{\mathfrak{D}} \hat{\otimes} \bar{\mathcal{T}}^{\mathfrak{D}}$. One can see it as a character on $\hat{\mathcal{T}}_-^{\mathfrak{D}} \hat{\otimes} \hat{\mathcal{T}}_+^{\mathfrak{D}}$ if one extends the definition of $\hat{\Pi}_x$ to $\hat{\mathcal{T}}_+^{\mathfrak{D}}$ on which is now an algebra. Then, we consider the following coactions $\hat{\Delta}^- : \hat{\mathcal{T}}_+^{\mathfrak{D}} \to \bar{\mathcal{T}}_-^{\mathfrak{D}} \otimes \hat{\mathcal{T}}_+^{\mathfrak{D}}$ and $\hat{\Delta}^+ : \hat{\mathcal{T}}_+^{\mathfrak{D}} \to \hat{\mathcal{T}}_+^{\mathfrak{D}} \otimes \bar{\mathcal{T}}_+^{\mathfrak{D}}$.

This Birkhoff factorisation happens on the Hopf algebra $\bar{\mathcal{T}}_-^{\mathfrak{D}} \hat{\otimes} \bar{\mathcal{T}}_+^{\mathfrak{D}}$ introduced in [BHZ19, Section 3.8]. The unit (counit) is given by $\mathbf{1}_1 \hat{\otimes} \mathbf{1}$ (resp. $\mathbf{1}_1^* \hat{\otimes} \mathbf{1}^*$). The coproduct $\Delta_\pm$ and the product are defined by:

$$\begin{aligned}(a \otimes b) \cdot (\bar{a} \otimes \bar{b}) &= (a \cdot \bar{a}) \otimes (b \cdot \bar{b}) , \\ \Delta_\pm(a \otimes b) &= \mathcal{M}^{(14)(3)(2)(5)}(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \bar{\Delta}^-)(\bar{\Delta}^- \otimes \bar{\Delta}^+)(a \otimes b) .\end{aligned} \tag{4.22}$$

One can also define the antipode $\mathcal{A}_\pm$:

$$\mathcal{A}_\pm = (\mathcal{A}_- \mathcal{M} \otimes \mathcal{A}_+)(\mathrm{id} \otimes \bar{\Delta}^-)$$

where $\bar{\Delta}^-$ is extended as a map from $\bar{\mathcal{T}}_+^{\mathfrak{D}}$ into $\bar{\mathcal{T}}_-^{\mathfrak{D}} \otimes \bar{\mathcal{T}}_+^{\mathfrak{D}}$ and the twisted antipode:

$$\tilde{\mathcal{A}}_\pm = (\tilde{\mathcal{A}}_- \mathcal{M} \otimes \tilde{\mathcal{A}}_+)(\mathrm{id} \otimes \Delta^-)$$

The coaction $\hat{\Delta}_\pm$ and the reduced coaction $\tilde{\Delta}_\pm$ are given by

$$\begin{aligned}\hat{\Delta}_\pm &= \mathcal{M}^{(14)(3)(2)(5)}(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \hat{\Delta}^-)(\hat{\Delta}^- \otimes \hat{\Delta}^+) \\ \tilde{\Delta}_\pm &= \mathcal{M}^{(14)(3)(2)(5)}(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \tilde{\Delta}^-)(\tilde{\Delta}^- \otimes \tilde{\Delta}^+).\end{aligned}$$

4.3 Numerical schemes for dispersive PDEs

In [BS20], the authors derived a resonance based numerical schemes for equations of the type

$$\begin{aligned} & i\partial_t u(t,x) + \mathscr{L}\left(\nabla, \frac{1}{\varepsilon}\right) u(t,x) = |\nabla|^\alpha p(u(t,x), \overline{u}(t,x)) \\ & u(0,x) = v(x), \quad (t,x) \in \mathbf{R}_+ \times \mathbf{T}^d \end{aligned} \tag{4.23}$$

where $\mathscr{L}$ is a differential operator and p is a polynomial nonlinearity of the form $p(u,\bar{u}) = u^N \bar{u}^M$. They approximate in Fourier space the iterated integrals obtained from a Duhamel's formula:

$$\hat{u}_k(t) = e^{itP(k)} \hat{v}_k - i |\nabla|^\alpha (k) \, e^{itP(k)} \int_0^t e^{-i\xi P(k)} p_k(u(\xi), \bar{u}(\xi)) d\xi \tag{4.24}$$

where $P(k)$ denotes the differential operator $\mathscr{L}$ in Fourier space

$$P(k) = \mathscr{L}\left(\nabla, \frac{1}{\varepsilon}\right)(k)$$

and

$$p_k(u(t), \bar{u}(t)) = \sum_{k = \sum_i k_i - \sum_j \bar{k}_j} \prod_{i=1}^N \hat{u}_{k_i}(t) \prod_{j=1}^M \bar{\hat{u}}_{\bar{k}_j}(t).$$

In order to encode these integrals, one needs decorations for the k, e^{itP} and polynomials in ξ. In fact, one considers a space of decorated forests denoted by $\mathscr{F}_+$ endowed with a structure of connected Hopf algebra with coproduct Δ^+. The authors also introduce a right-comodule $\mathscr{F}$ over $\mathscr{F}_+$ with a coaction $\Delta : \mathscr{F} \to \mathscr{F} \otimes \mathscr{F}_+$. Then, the main maps for computing the approximations are given by a family of characters $(\Pi_n)_{n \in \mathbf{N}}$, $\Pi_n : \mathscr{F} \to \mathscr{C}$ where n corresponds to the regularity assumed a priori on the solution. The definition of the characters Π_n is close in spirit to the recursive definition given in (4.2) for $\mathbf{\Pi}$. The main difference happens for the integrals $\int_0^t e^{-i\xi P(k)} \cdots d\xi$ which need to be approximated through a well-chosen Taylor expansion depending on the resonances. Indeed, one Talor-expands the lower part in the resonances and integrates exactly the dominant part. This will guarantee a minimisation of the local error.

The target space $\mathscr{C}$ consists of trigonometric polynomials which are functions of the form $z \mapsto \sum_j Q_j(z) e^{izP_j(k_1,\dots,k_n)}$, where the $Q_j(z)$ are polynomials in z and the P_j are polynomials in $k_1, ..., k_n \in \mathbf{Z}^d$. Endowed with the pointwise product, $\mathscr{C}$ is an algebra. Then, one can give the following splitting:

$$\mathscr{C} = \mathscr{C}_- \oplus \mathscr{C}_+, \quad \mathbb{Q} : \mathscr{C} \to \mathscr{C}_-,$$

where $\mathscr{C}_-$ is the space of polynomials $Q(\xi)$ and $\mathscr{C}_+$ is the subspace of trigonometric polynomials which are elements of the form $z \mapsto \sum_j Q_j(z) e^{izP_j(k_1,\ldots,k_n)}$ with $P_j \neq 0$. Then, one constructs another character $\hat{\Pi}_n$ such that

$$\hat{\Pi}_n = (\Pi_n \otimes (\mathbb{Q} \circ \Pi_n \mathscr{A} \cdot)(0))\Delta. \tag{4.25}$$

where $\mathscr{A}$ is the antipode associated to $\mathscr{F}_+$, Π_n is extended as a character onto $\mathscr{F}_+$ and $\mathbb{Q} \circ \Pi_n \mathscr{A}$ is defined multiplicatively for two decorated forests τ and $\bar{\tau}$ by:

$$(\mathbb{Q} \circ \Pi_n \mathscr{A})(\tau \cdot \bar{\tau}) := (\mathbb{Q} \circ \Pi_n \mathscr{A})(\tau)(\mathbb{Q} \circ \Pi_n \mathscr{A})(\bar{\tau}).$$

If one considers the space of the symmetrised tensor $\mathfrak{N} = T(\mathscr{C})$, then

$$T(\mathscr{C}) = T(\mathscr{C}_+) \oplus \mathscr{C}_-, \quad \tilde{\mathbb{Q}} : T(\mathscr{C}) \to \mathscr{C}_-,$$

where the projection $\tilde{\mathbb{Q}}$ is defined as

$$\tilde{\mathbb{Q}}(\widehat{f} \otimes \widehat{g}) = \tilde{\mathbb{Q}}(\widehat{f})\tilde{\mathbb{Q}}(\widehat{g}), \quad \widehat{f}, \widehat{g} \in T(\mathscr{C}), \quad \tilde{\mathbb{Q}}(f) = \mathbb{Q}(f), \quad f \in \mathscr{C}.$$

Then, we define Π_n as a character from $\mathscr{F}$ into $T(\mathscr{C})$ by replacing the pointwise product by the tensor product. We denote this new character as $\tilde{\Pi}_n$ and one gets:

$$\hat{\Pi}_n = \Big(\tilde{\Pi}_n \otimes (\tilde{\mathbb{Q}} \circ \tilde{\Pi}_n \mathscr{A} \cdot)(0)\Big)\Delta,$$

where now $\hat{\Pi}_n$ is viewed as a character taking values in $T(\mathscr{C})$. This framework matches the the one mentioned in Remark 2.10, where the map Q is equal to $\mathrm{ev}_0 \circ \tilde{\mathbb{Q}}$. The character $\hat{\Pi}_n$ is crucial for better understanding the error introduced by the character Π_n. It allows to derive one of the main theorem for the local error analysis of the scheme see [BS20, Theorem 3.11]. The idea behind its construction is that we want to single out oscillations that's why the projector $\tilde{\mathbb{Q}}$ is applied repeatedly in the recursion.

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